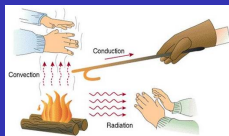


Heat and Mass Transfer



Introduction

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Prerequisites

Thermodynamics, Fluid Mechanics

References

- Incropera FP and Dewitt DP, *Fundamentals of Heat and Mass Transfer*, Fifth edition, John Wiley and Sons, 2010.
- Cengel YA, *Heat and Mass Transfer - A Practical Approach*, Third edition, McGraw-Hill, 2010.
- Holman JP, *Heat Transfer*, McGraw-Hill, 1997.

Class Timings: ME305

Tue: 9 AM to 10 AM, Room-107

Wed, Thu, Fri: 11 AM to 12 AM, Room-107 **Weblinks**

www.iitp.ac.in/~sudheer/teaching.html



Introduction:

What, How, and Where?

Thermodynamics and Heat transfer

Application

Physical mechanism of heat transfer

Conduction:

Introduction

1D, steady-state

2D, steady-state

Transient

Convection:

Introduction

External and internal flows

Free convection

Boiling and condensation

Heat exchangers

Radiation:

Introduction

View factors

Mass Transfer:

Introduction

Mass diffusion equation

Transient diffusion



The science that deals with the determination of the rates of energy transfer due to temperature difference.

Driving force

Temperature difference

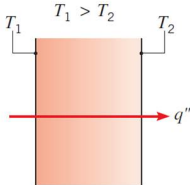
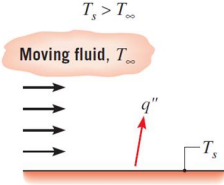
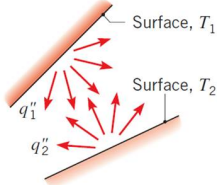
as the voltage difference in electric current

as the pressure difference in fluid flow

Rate depends on magnitude of dT

Heat Transfer - How?



Conduction through a solid or a stationary fluid	Convection from a surface to a moving fluid	Net radiation heat exchange between two surfaces
		

Mundra Thermal Power Plant, Gujrat 4620 MW



Heat Transfer - Where else?





Thermodynamics

Deals with the amount of energy (heat or work) during a process
Only considers the end states in equilibrium
Why?

Heat Transfer

Deals with the rate of energy transfer
Transient and non-equilibrium
How long?



Laws of Thermodynamics

Zeroth law - Temperature

First law Energy conserved

Second law Entropy

Third law $S \rightarrow \text{constant}$ as $T \rightarrow 0$

Laws of Heat Transfer

Fouriers law - Conduction

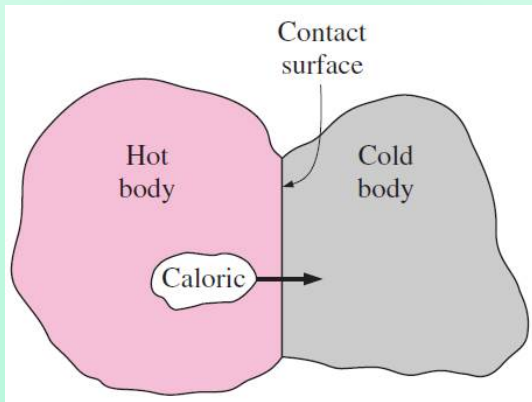
Newtons law of cooling - Convection

Stephan-Boltzmann law - Radiation

Caloric theory (18th Century)

Heat is a fluid like substance, '*caloric*' poured from one body into another.

Caloric: Massless, colorless, odorless, tasteless





Kinetic theory (19th Century)

Molecules - tiny balls - are in motion possessing kinetic energy

Heat: The energy associated with the random motion of atoms and molecules



Heat

The amount of heat transferred during a process, Q

Heat transfer rate

The amount of heat transferred per unit time, \dot{Q} or simply q

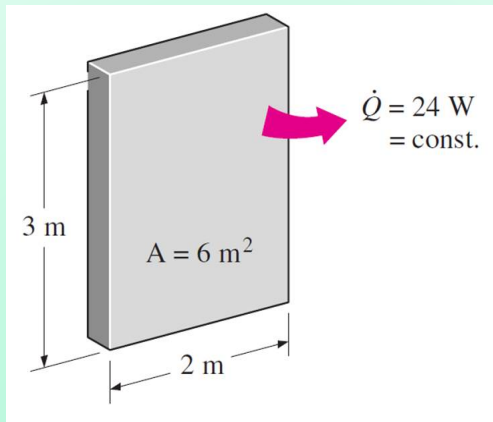
$$Q = \int_0^{\Delta t} q dt$$

$$Q = q\Delta t, \text{ if } q \text{ is constant}$$

Heat flux

The rate of heat transfer per unit area normal to the direction of heat transfer:

$$q'' = \frac{q}{A}$$

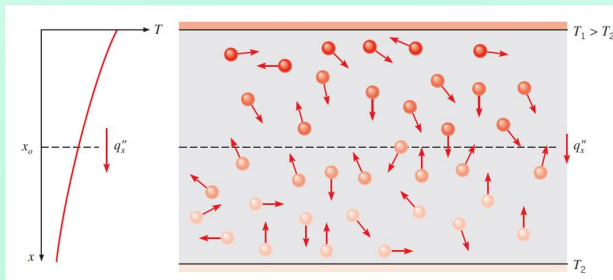


$$q'' = \frac{24 \text{ W}}{6 \text{ m}^2} = 4 \text{ W/m}^2$$

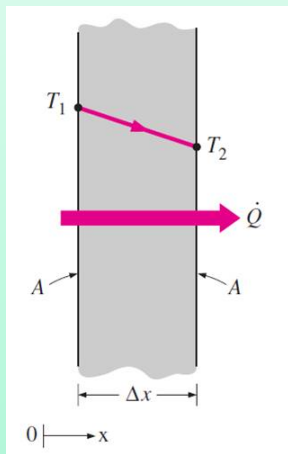
Viewed as

The transfer of energy from the more energetic to the less energetic particles of a substance due to interactions between the particles.

Net transfer by random molecules motion - *diffusion of energy*



Conduction: Fourier's Law of Heat Conduction



$$q_{cond} = -kA \frac{T_1 - T_2}{\Delta x} = -kA \frac{dT}{dx}$$



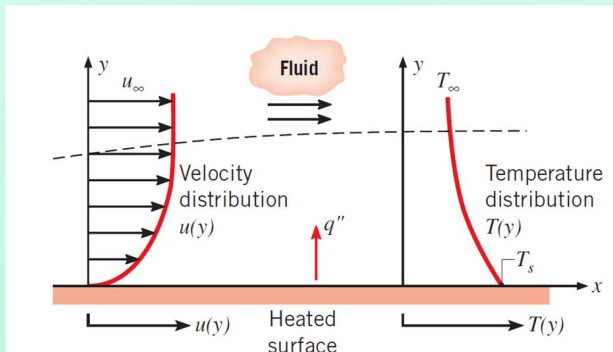
The wall of an industrial furnace is constructed from 0.15 m thick fireclay brick having a thermal conductivity of 1.7 W/m K. Measurements made during steady-state operation reveal temperatures of 1400 and 1150 K at the inner and outer surfaces, respectively. What is the rate of heat loss through a wall that is $0.5 \times 1.2 \text{ m}^2$ on a side?

Ans: 1.7 kW

Comprised of two mechanisms

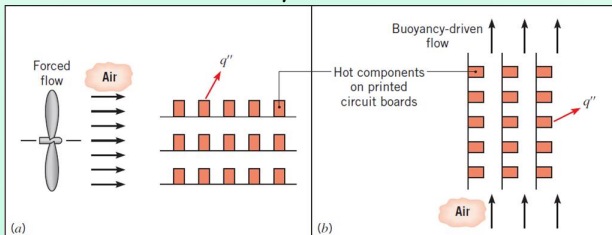
Energy transfer due to random molecular motion - *diffusion*

Energy transfer by the bulk motion of the fluid - *advection*

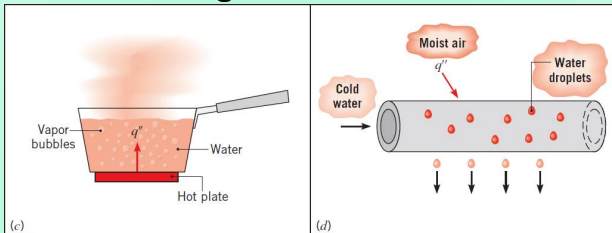


Boundary layer development in convection heat transfer

Forced and Free/Natural Convection



Boiling and Condensation





$$q_{conv} = hA_s (T_s - T_\infty)$$

Process	h (W/m ² K)
Free convection	
Gases	2-25
Liquids	50-1000
Convection with phase change Boiling and Condensation	2500-100,000



Radiation

Energy emitted by matter that is at a nonzero temperature

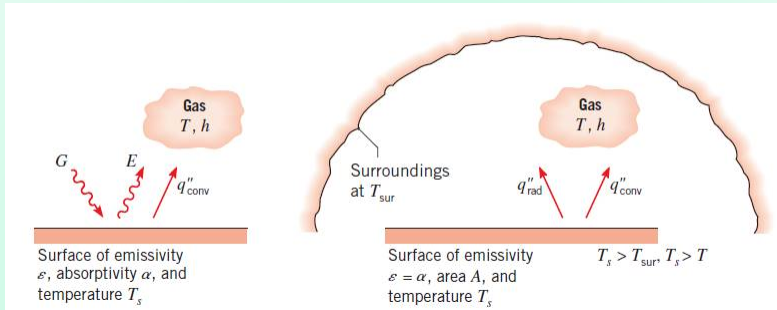
Transported by electromagnetic waves (or photons)

Medium?

Surface Emissive Power

The rate at which energy is released per unit area (W/m^2)

$$E_b = \sigma T_s^4$$

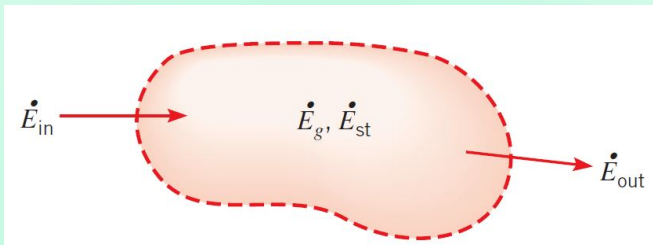


For a real surface:

$$E = \varepsilon \sigma T_s^4$$
$$q''_{\text{rad}} = \varepsilon \sigma (T_s^4 - T_{\text{sur}}^4)$$

$$E_{in} - E_{out} = \Delta E_{st}$$

In rate form:



$$\dot{E}_{in} - \dot{E}_{out} = \frac{dE_{st}}{dt} = \dot{E}_{st}$$



The inflow and outflow terms are *surface phenomena*.

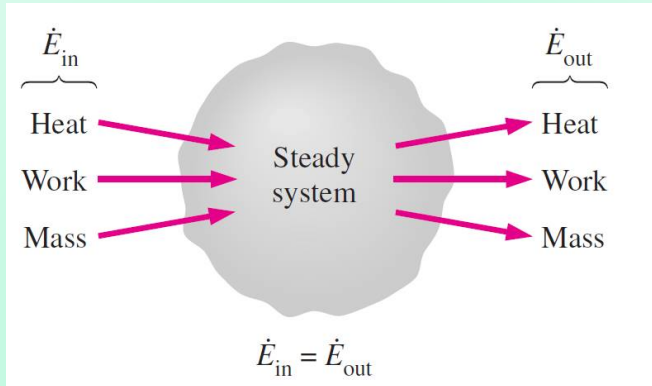
The *energy generation term* is a volumetric phenomenon.
chemical, electrical

The *energy storage* is also a volumetric phenomenon.

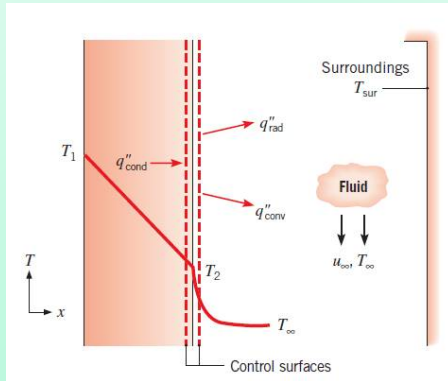
$$\Delta U + \Delta KE + \Delta PE$$

ΔU : sensible/thermal, latent, and chemical components

Steady state with no heat generation



Surface Energy Balance



$$E_{in} - E_{out} = 0$$
$$q_{cond} - q_{conv} - q_{rad} = 0$$



Analysis of different problems will give a deeper appreciation for the fundamentals of the subject, and you will gain confidence in your ability to apply these fundamentals to the solution of engineering problems.

Be consistent in following these steps:

- 1 known
- 2 Find
- 3 Schematic
- 4 Assumptions
- 5 Properties
- 6 Analysis
- 7 Comments



The hot combustion gases of a furnace are separated from the ambient air and its surrounding, which are at 25°C , by a brick wall 0.15 m thick. The brick has a thermal conductivity of 1.2 W/m K and a surface emissivity of 0.8. Under steady-state conditions an outer surface temperature of 100°C is measured. Free convection heat transfer to the air adjoining the surface is characterized by a convection coefficient of $20 \text{ W/m}^2 \text{ K}$. What is the brick inner surface temperature.

Ans: 625 K



An experiment to determine the convection coefficient associated with airflow over the surface of a thick stainless steel casting involves the insertion of thermocouples into the casting at distances of 10 and 20 mm from the surface along a hypothetical line normal to the surface. The steel has a thermal conductivity of 15 W/m K. If the thermocouples measure temperatures of 50 and 40°C in the steel when the air temperature is 100°C, what is the convection coefficient?

Ans: 375 W/m² K

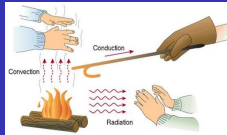


The roof of a car in a parking lot absorbs a solar radiant flux of 800 W/m^2 , and the underside is perfectly insulated. The convection coefficient between the roof and the ambient air is $12 \text{ W/m}^2 \text{ K}$.

- Neglecting radiation exchange with the surroundings, calculate the temperature of the roof under steady-state conditions if the ambient air temperature is 20°C .
- For the same ambient air temperature, calculate the temperature of the roof if its surface emissivity is 0.8.
- The convection coefficient depends on air flow conditions over the roof, increasing with increasing air speed. Compute and plot the roof temperature as a function of h for $2 \leq h \leq 200 \text{ W/m}^2 \text{ K}$.

Ans: 86.7°C

Heat and Mass Transfer



Heat Diffusion Equation

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Fourier's law of heat conduction

$$q_{cond} = -kA \frac{dT}{dx}$$

transient

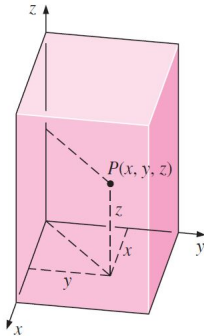
multidimensional - complex geometries

Steady-state heat transfer

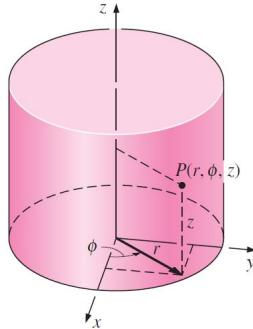
- No change with time at any point within the medium
- T and q'' remains unchanged with time
- $T = T(x, y, z)$
- Usually no but assumed

Transient heat transfer

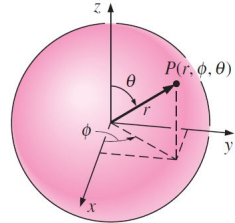
- Time dependence
- $T = T(x, y, z, t)$
- Special case - *lumped* - T changes with time but not with location:
 $T = T(t)$



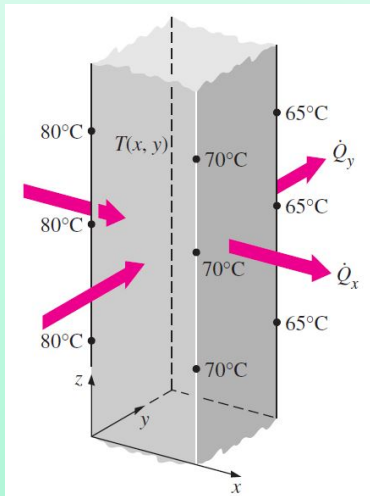
(a) Rectangular coordinates



(b) Cylindrical coordinates



(c) Spherical coordinates



Heat Flux Direction

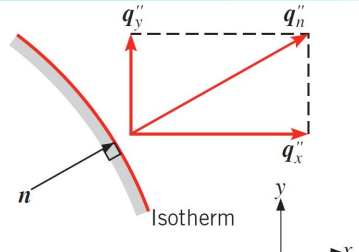
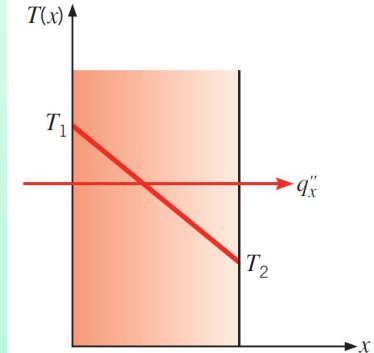
The direction of heat flow will always be normal to a surface of constant temperature, called an isothermal surface.

$$q''_x = -k \frac{\partial T}{\partial x}; q''_y = -k \frac{\partial T}{\partial y}; q''_z = -k \frac{\partial T}{\partial z}$$

$$\begin{aligned} q''_n &= q''_x \vec{i} + q''_y \vec{j} + q''_z \vec{k} \\ &= -k \left(\frac{\partial T}{\partial x} \vec{i} + \frac{\partial T}{\partial y} \vec{j} + \frac{\partial T}{\partial z} \vec{k} \right) \\ &= -k \nabla T \end{aligned}$$

where n is the normal of the isothermal surface and

$$q''_n = -k \frac{\partial T}{\partial n}$$



Thermal conductivity

$$k = \frac{q''}{(\partial T / \partial x)}$$

The rate of heat transfer through a unit thickness of the material per unit area per unit temperature difference.

Specific heat, C_p

Ability to store thermal energy.
At room temperature,

$$\begin{aligned} C_p &= 4.18 \text{ kJ/kg K, water} \\ &= 0.45 \text{ kJ/kg K, iron} \end{aligned}$$

Thermal conductivity, k

Material's ability to conduct heat
At room temperature,

$$\begin{aligned} k &= 0.607 \text{ W/m K, water} \\ &= 80.2 \text{ W/m K, iron} \end{aligned}$$



- Transport property
- Indication of the rate at which energy is transferred by the diffusion process
- Depends on the physical structure of matter, atomic and molecular, related to the state of the matter
- *Isotropic* material - k is independent of the direction of transfer, $k_x = k_y = k_z$

Laminated composite materials and wood

k across grain is different than that parallel to grain

k for Different Materials at T_∞ and P_∞



Kinetic theory of gases:

$$v_{rms} = \sqrt{\frac{3RT}{M}}$$

$$T \uparrow \quad k \uparrow$$

$$M \uparrow \quad k \downarrow$$

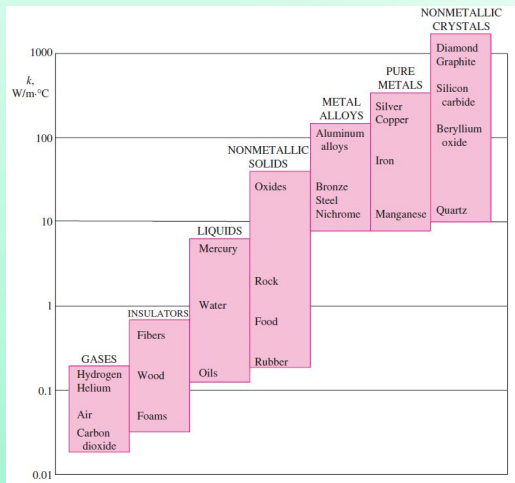
He(4), Air(29)

Liquids: Strong intermolecular forces

$$\text{Most liquids: } T \uparrow \quad k \downarrow$$

$$M \uparrow \quad k \downarrow$$

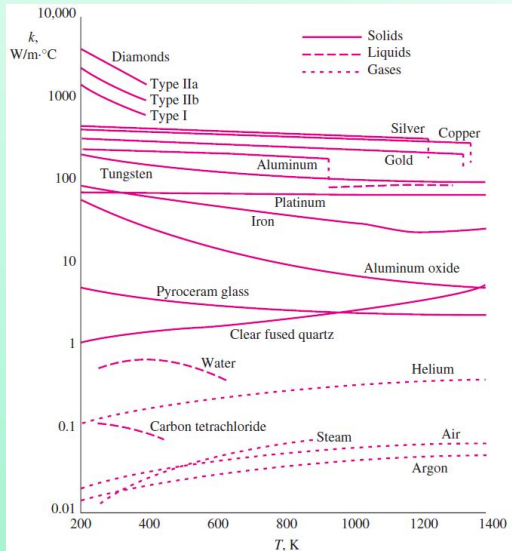
Except water: Not a linear trend



k - Temp. Dependency

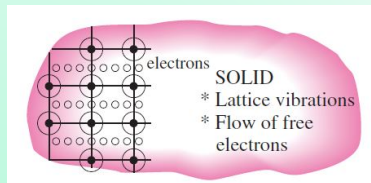


- Temp. dependency causes considerable complexity in conduction analysis
- $k_{average}$



$$k = k_l + k_e$$

High k for pure metals is due to k_e
 k_l depends on the way the molecules are arranged



Diamond - highly ordered crystalline solid

Highest known k

However a poor electric conductor (even semiconductors like silicon)

Diamond heat sinks - cooling electronic components



Pure metals have high k

$$k_{iron} = 83 \text{ W/m K}$$

$$k_{chromium} = 95 \text{ W/m K}$$

Steel is iron + 1% chrome

$$k_{steel} = 62 \text{ W/m K}$$

Alloy of two metals k_1 and $k_2 < k_1$ and k_2



- Thermophysical properties
 - k Transport property
 - ρ, C_p Thermodynamic properties
- ρC_p is volumetric heat capacity ($\text{J}/\text{m}^3 \text{K}$)

- High α : faster propagation of heat into the medium
- Small α : heat is mostly absorbed by the material and a small amount of heat is conducted further



Problem and application

- Determine *temperature distribution* in a medium resulting from conditions imposed on its boundaries
- The conduction heat flux at any point in the medium or on the surface may be computed from Fourier's law
- This information could be used to determine thermal stresses, expansions, deflections
- Temperature distribution may also be used to optimize the thickness of an insulating material or to determine the compatibility of special coatings or adhesives used with the material

Assumptions

Homogeneous medium

No bulk motion (advection)

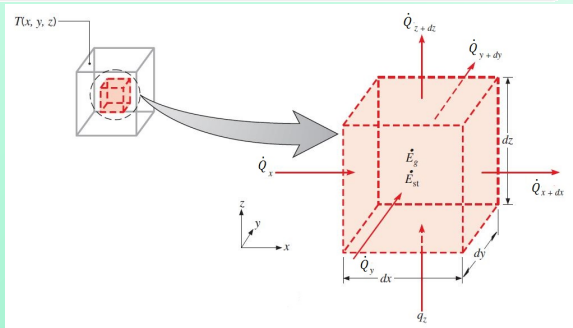
Schematic

Consider an infinitesimally small (differential) CV, $dx \cdot dy \cdot dz$

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx$$

$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y} dy$$

$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz$$





Generation

$$\dot{E}_g = \dot{e}_g dx dy dz$$

\dot{e}_g is in W/m^3

Storage

$$\dot{E}_{st} = \underbrace{\rho C_p \frac{\partial T}{\partial t}}_{\downarrow} dx dy dz$$

Rate of change of the sensible/thermal energy of the medium/volume

Governing Equation

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$

$$q_x + q_y + q_z - q_{x+dx} - q_{y+dy} - q_{z+dz} + \dot{e}_g dx dy dz = \rho C_p \frac{\partial T}{\partial t} dx dy dz$$
$$-\frac{\partial q_x}{\partial x} dx - \frac{\partial q_y}{\partial y} dy - \frac{\partial q_z}{\partial z} dz + \dot{e}_g dx dy dz = \rho C_p \frac{\partial T}{\partial t} dx dy dz$$

However,

$$q_x = -k dy dz \frac{\partial T}{\partial x}; \quad q_y = -k dx dz \frac{\partial T}{\partial y}; \quad q_z = -k dx dy \frac{\partial T}{\partial z}$$

$$\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{e}_g = \rho C_p \frac{\partial T}{\partial t}$$



Fourier-Biot equation - Isotropic

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_g}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

Diffusion equation - Transient, no heat generation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

Poisson equation - Steady-state

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_g}{k} = 0$$

Laplace equation - Steady-state, no heat generation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

Cartesian coordinates $T(x, y, z)$

$$\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{e}_g = \rho C_p \frac{\partial T}{\partial t}$$

Cylindrical coordinates $T(r, \phi, z)$

$$\frac{1}{r} \left(kr \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \phi} \left(k \frac{\partial T}{\partial \phi} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{e}_g = \rho C_p \frac{\partial T}{\partial t}$$

Spherical coordinates $T(r, \phi, \theta)$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(kr^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left(k \frac{\partial T}{\partial \phi} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(k \sin \theta \frac{\partial T}{\partial \theta} \right) + \dot{e}_g = \rho C_p \frac{\partial T}{\partial t}$$



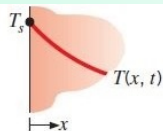
- Necessary to solve the appropriate form of the heat equation
 - Depends on the physical conditions at boundaries
 - On time
- Boundary conditions can be simply expressed in mathematical form
 - Second order in space, two boundary conditions for each coordinate needed to describe the system
 - First order in time, only one condition, *initial condition* must be specified

Boundary Conditions at $x = 0$



1. Constant surface temperature

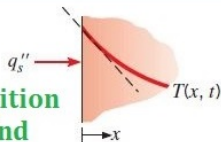
$$T(0, t) = T_s \quad \text{Dirichlet Condition} \\ \text{BC of first kind}$$



2. Constant surface heat flux

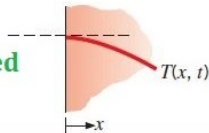
- (a) Finite heat flux

$$-k \frac{\partial T}{\partial x} \Big|_{x=0} = q_s'' \quad \text{Neumann Condition} \\ \text{BC of second kind}$$



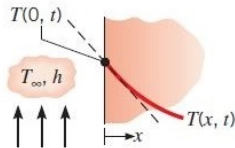
- (b) Adiabatic or insulated surface

$$\frac{\partial T}{\partial x} \Big|_{x=0} = 0 \quad \text{Perfectly insulated} \\ \text{Adiabatic}$$



3. Convection surface condition

$$-k \frac{\partial T}{\partial x} \Big|_{x=0} = h[T_\infty - T(0, t)] \\ \text{BC of third kind}$$





A long copper bar of rectangular cross section, whose width w is much greater than its thickness L , is maintained in contact with a heat sink at its lower surface, and the temperature throughout the bar is approximately equal to that of the sink, T_o . Suddenly, an electric current is passed through the bar and an airstream of temperature T_∞ is passed over the top surface, while the bottom surface continues to be maintained at T_o . Obtain the differential equation and the boundary and initial conditions that could be solved to determine the temperature as a function of position and time in the bar.

Solution: Diffusion Equation

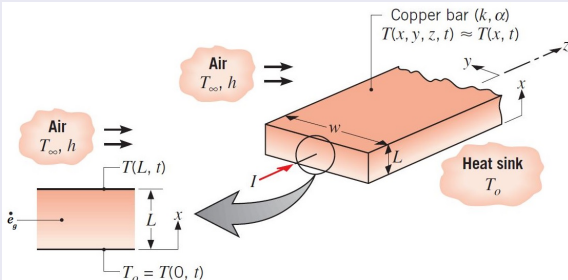
Known

- Copper bar initial temperature, T_o
- Suddenly heated up by electric current, \dot{e}_g
- Airstream, h, T_∞

Find

Differential equation;
Boundary conditions;
Initial condition

Schematic





Assumptions

- $w \ll L$ - side effects are neglected. Thus heat transfer is primarily one dimensional (x)
- Uniform volumetric heat generation, \dot{e}_g
- Constant properties

Analysis

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{e}_g}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

Boundary conditions

$$T(0, t) = T_o$$

$$-k \frac{\partial T}{\partial x} \Big|_{x=L} = h [T(L, t) - T_\infty]$$

Initial condition

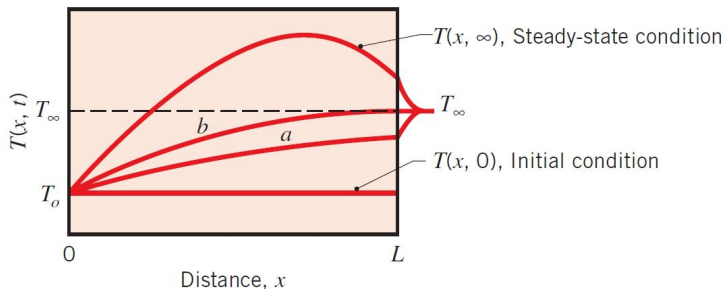
$$T(x, 0) = T_o$$

Solution: Diffusion Equation

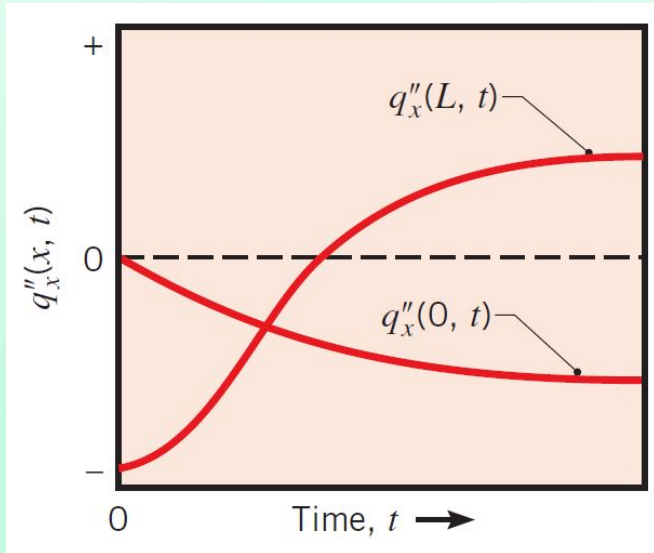


Comments

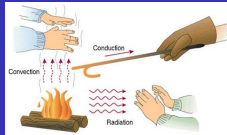
- 1 If T_o, T_∞, \dot{e}_g , and h are known, then the equations can be solved to obtain the $T(x, t)$
- 2 Top surface, $T(L, t)$ will change with time. This is unknown and may be obtained after finding $T(x, t)$



Solution: Diffusion Equation



Heat and Mass Transfer



One-Dimensional, Steady-State Conduction

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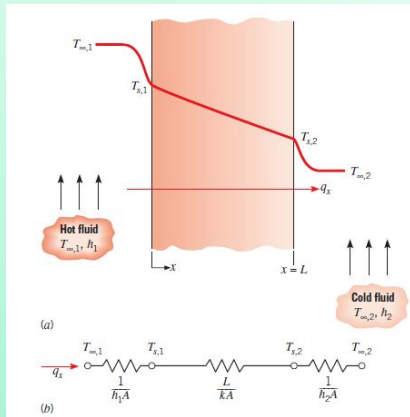
- Temp. gradients exist along only a single coordinate direction
- Heat transfer occurs exclusively in that direction
- Temp. at each point is independent of time

We will see:

- Temp. distribution & heat transfer rate in common (planar, cylindrical and spherical) geometries
- Thermal resistance
 - Thermal circuits to model heat flow
 - Electrical circuits to current flow

$$\frac{d}{dx} \left(k \frac{dT}{dx} \right) = 0$$

For 1-D, steady-state conduction in a plane wall with no heat generation, heat flux is a constant, independent of x .





If k is constant then, $T(x) = C_1x + C_2$

$$T(0) = T_{s,1} \quad \text{and} \quad T(L) = T_{s,2}$$

$$T(x) = (T_{s,2} - T_{s,1}) \frac{x}{L} + T_{s,1}$$

$$q_x = -kA \frac{dT}{dx} = \frac{kA}{L} (T_{s,2} - T_{s,1})$$

$$q_x'' = \frac{k}{L} (T_{s,2} - T_{s,1})$$

Ratio of driving potential to the corresponding transfer rate

$$R_{t,cond} = \frac{(T_{s,1} - T_{s,2})}{q_x} = \frac{L}{kA}$$

$$R_e = \frac{E_{s,1} - E_{s,2}}{I}$$

$$R_{t,conv} = \frac{(T_s - T_\infty)}{q} = \frac{1}{hA}$$

Under steady state condi-

tions:

Convection rate into the wall	=	Conduction rate through the wall	=	Convection rate from the wall
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$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2A}$$

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}} \quad R_{tot} = \frac{1}{h_1A} + \frac{L}{kA} + \frac{1}{h_2A}$$



The **thermal resistance for radiation** - radiation exchange between the surface and its surroundings:

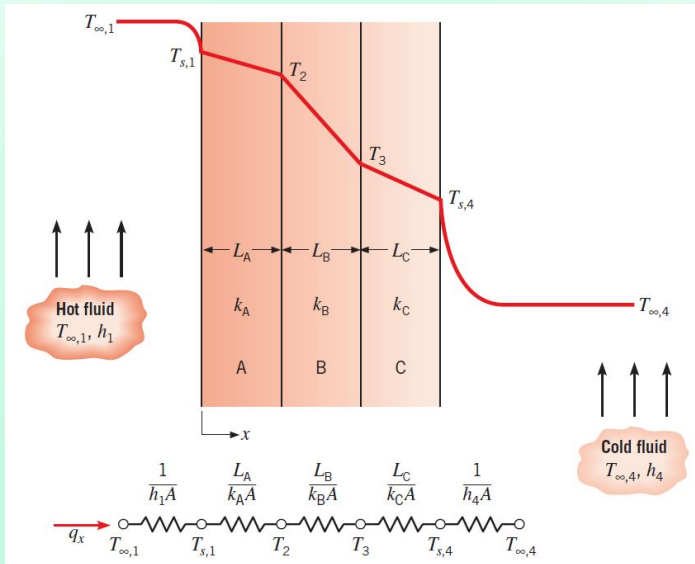
$$R_{t,rad} = \frac{T_s - T_{sur}}{q_{rad}} = \frac{1}{h_r A}$$

$$q_{rad} = h_r A (T_s - T_{sur})$$

The radiation heat transfer coefficient, h_r :

$$h_r = \varepsilon \sigma (T_s + T_{sur}) (T_s^2 + T_{sur}^2)$$

The Composite Wall





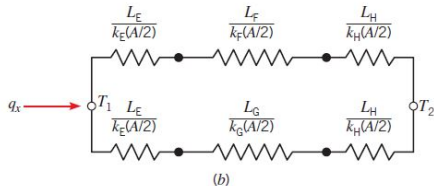
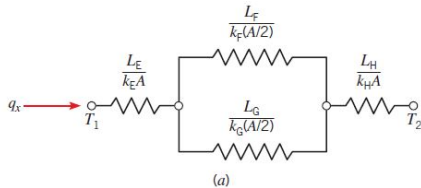
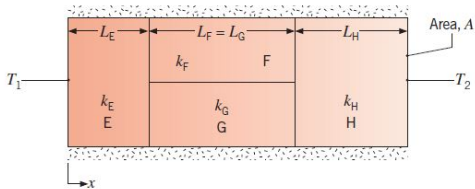
$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\sum R_t}$$
$$R_{tot} = \frac{1}{h_1 A} + \frac{L_A}{k_A A} + \frac{L_B}{k_B A} + \frac{L_C}{k_C A} + \frac{1}{h_4 A}$$

If U is the **overall heat transfer coefficient**

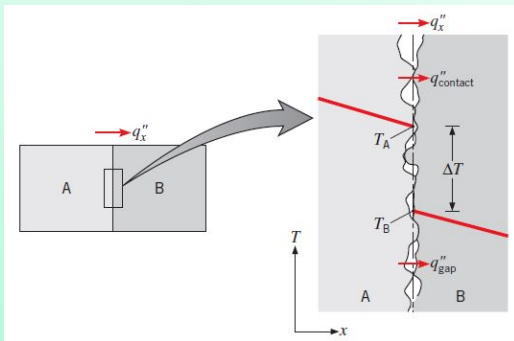
$$q_x = U A \Delta T$$

$$U = \frac{1}{R_{tot} A}$$

Series-Parallel Composite Wall



$$R''_{t,c} = \frac{T_A - T_B}{q''_x}$$



Thermal Resistance, $R''_{t,c} \times 10^4$ ($\text{m}^2 \cdot \text{K}/\text{W}$)

(a) Vacuum Interface

Contact pressure	100 kN/m^2	10,000 kN/m^2
Stainless steel	6–25	0.7–4.0
Copper	1–10	0.1–0.5
Magnesium	1.5–3.5	0.2–0.4
Aluminum	1.5–5.0	0.2–0.4

(b) Interfacial Fluid

Air	2.75
Helium	1.05
Hydrogen	0.720
Silicone oil	0.525
Glycerine	0.265

Thermal Resistance of Solid/Solid Interfaces



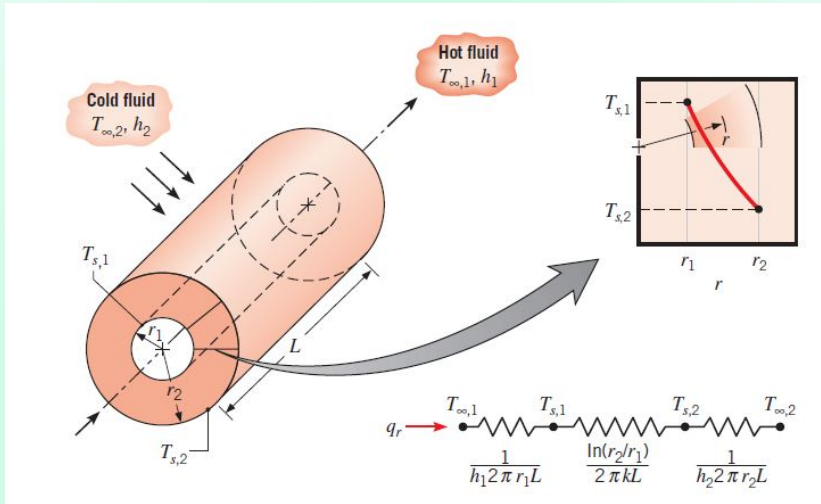
Interface	$R''_{t,c} \times 10^4 \text{ (m}^2 \cdot \text{K/W)}$
Silicon chip/lapped aluminum in air (27–500 kN/m ²)	0.3–0.6
Aluminum/aluminum with indium foil filler ($\sim 100 \text{ kN/m}^2$)	~ 0.07
Stainless/stainless with indium foil filler ($\sim 3500 \text{ kN/m}^2$)	~ 0.04
Aluminum/aluminum with metallic (Pb) coating	0.01–0.1
Aluminum/aluminum with Dow Corning 340 grease ($\sim 100 \text{ kN/m}^2$)	~ 0.07
Stainless/stainless with Dow Corning 340 grease ($\sim 3500 \text{ kN/m}^2$)	~ 0.04
Silicon chip/aluminum with 0.02-mm epoxy	0.2–0.9
Brass/brass with 15- μm tin solder	0.025–0.14



For a porous medium, an effective conductivity is considered. Assuming, there is no fluid bulk motion and if $T_1 > T_2$

$$q_x = \frac{k_{eff}A}{L}(T_1 - T_2)$$

The Cylinder





The governing equation for 1D, steady state conduction in cylindrical coordinates:

$$\frac{1}{r} \frac{d}{dr} \left(kr \frac{dT}{dr} \right) = 0$$

The heat flux by Fourier's law of conduction,

$$q_r = -kA \frac{dT}{dr} = -k(2\pi rL) \frac{dT}{dr}$$

- Here, $A = 2\pi rL$ is the area normal to the direction of heat transfer.
- The quantity $\frac{d}{dr} \left(kr \frac{dT}{dr} \right)$ is independent of r
- The conduction heat transfer rate q_r (not the heat flux, q_r'') is a constant in the radial direction



Temperature distribution and heat transfer rate

$$T(r) = \frac{T_{s,1} - T_{s,2}}{\ln(r_1/r_2)} \ln\left(\frac{r}{r_2}\right) + T_{s,2}$$

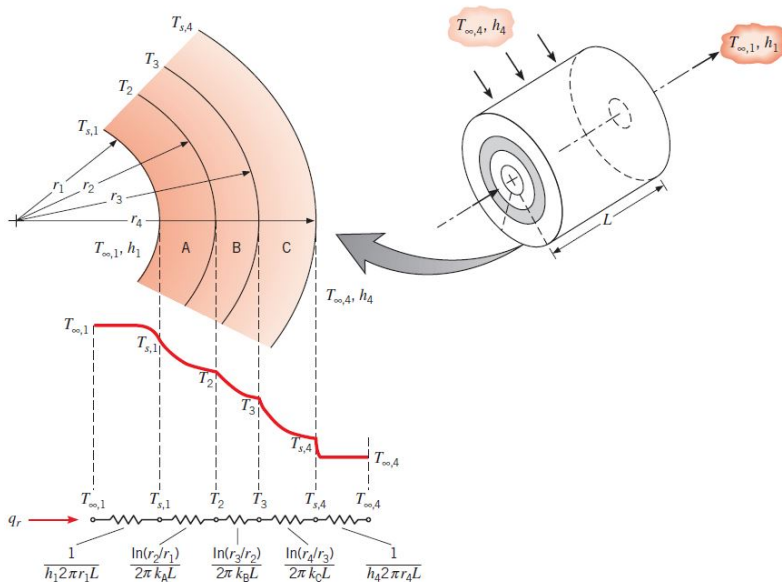
Note that the temperature distribution associated with radial conduction through a cylindrical wall is logarithmic, not linear, as it is for the plane wall.

$$q_r = \frac{2\pi Lk(T_{s,1} - T_{s,2})}{\ln(r_2/r_1)}$$

Note that q_r is independent of r .

$$R_{t,cond} = \frac{\ln(r_2/r_1)}{2\pi Lk}$$

The Cylinder





$$q_r = \frac{(T_{\infty,1} - T_{\infty,2})}{\frac{1}{2\pi r_1 L h_1} + \frac{\ln(r_2/r_1)}{2\pi k_A L} + \frac{\ln(r_3/r_2)}{2\pi k_B L} + \frac{\ln(r_4/r_3)}{2\pi k_C L} + \frac{1}{2\pi r_4 L h_4}}$$
$$q_r = \frac{(T_{\infty,1} - T_{\infty,2})}{R_{tot}} = UA (T_{\infty,1} - T_{\infty,2})$$

If U is defined in terms of the inside area, $A_1 = 2\pi r_1 L$, then:

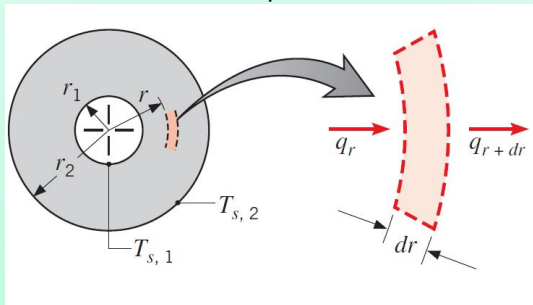
$$U = \frac{1}{\frac{1}{h_1} + \frac{r_1}{k_A} \ln\left(\frac{r_2}{r_1}\right) + \frac{r_1}{k_B} \ln\left(\frac{r_3}{r_2}\right) + \frac{r_1}{k_C} \ln\left(\frac{r_4}{r_3}\right) + \frac{r_1}{r_4} \frac{1}{h_4}}$$

- UA is constant, while U is not
- In radial systems, q is constant, while q'' is not

The Sphere



Consider a hollow sphere, whose inner and outer surfaces are exposed to fluids at different temperatures.



$$\frac{1}{r^2} \frac{d}{dr} \left(kr^2 \frac{dT}{dr} \right) = 0$$
$$q_r = -k(4\pi r^2) \frac{dT}{dr}$$



Temperature distribution and heat transfer rate

$$T(r) = T_{s,1} + \frac{T_{s,1} - T_{s,2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \left[\frac{1}{r_1} - \frac{1}{r} \right]$$

$$q_r = \frac{4\pi k (T_{s,1} - T_{s,2})}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$$

$$R_{t,cond} = \frac{1}{4\pi k} \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$



A spherical thin walled metallic container is used to store liquid nitrogen at 80 K. The container has a diameter of 0.5 m and is covered with an evacuated, reflective insulation composed of silica powder. The insulation is 25 mm thick, and its outer surface is exposed to ambient air at 310 K. The convection coefficient is known to be $20 \text{ W/m}^2 \text{ K}$. The latent heat of vaporization and the density of the liquid nitrogen are $2 \times 10^5 \text{ J/kg}$ and 804 kg/m^3 , respectively. Thermal conductivity of evacuated silica powder (300 K) is 0.0017 W/m K .

- 1 What is the rate of heat transfer to the liquid nitrogen?
- 2 What is the rate of liquid boil-off?

Solution: Sphere



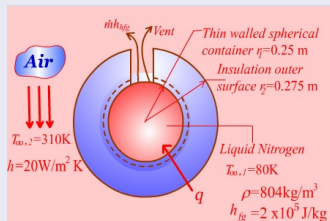
known

Liquid nitrogen is stored in a spherical container that is insulated and exposed to ambient air.

Find

- The rate of heat transfer to the nitrogen.
- The mass rate of nitrogen boil-off.

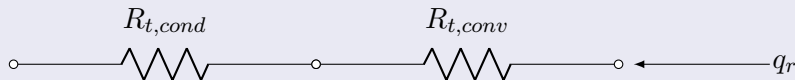
Schematic



Assumptions

- 1 Steady state conditions
- 2 One-dimensional transfer in the radial direction
- 3 Negligible resistance to heat transfer through the container wall and from the container to the nitrogen
- 4 Constant properties
- 5 Negligible radiation exchange between outer surface of insulation and surroundings

Analysis





$$R_{t,cond} = \frac{1}{4\pi k} \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$
$$R_{t,conv} = \frac{1}{h(4\pi r_2^2)} q_r = \frac{(T_{\infty,2} - T_{\infty,1})}{R_{t,cond} + R_{t,conv}} = 12.88W$$

Energy balance for a control surface about the nitrogen:

$$\dot{E}_{in} - \dot{E}_{out} = 0$$

$$q - \dot{m}h_{fg} = 0$$

$$\implies \dot{m} = 6.44 \times 10^{-5} \text{ kg/W}$$

$$= 5.56 \text{ kg/day}$$

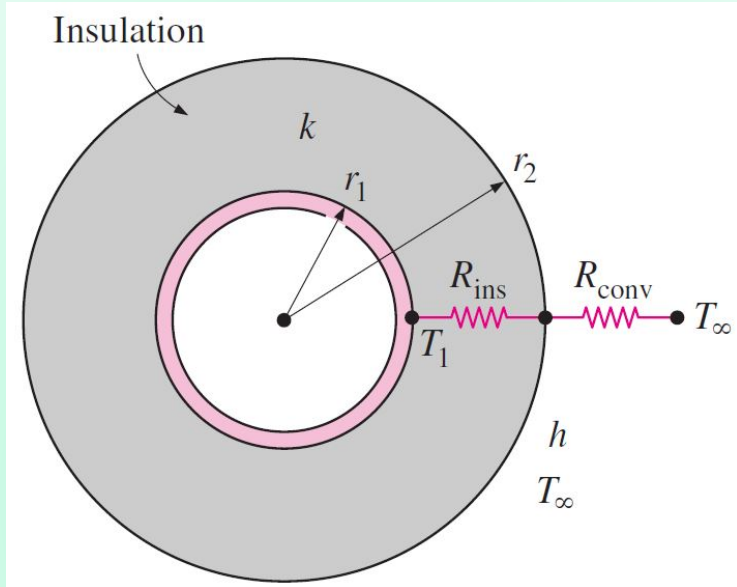
$$= \frac{\dot{m}}{\rho} = 0.00692 \text{ m}^3/\text{day}$$

$$= 6.92 \text{ liters/day}$$



- We know that by adding more insulation to a wall always decreases heat transfer.
- This is expected, since the heat transfer area A is constant, and adding insulation will always increase the thermal resistance of the wall without affecting the convection resistance.
- However, adding insulation to a cylindrical piece or a spherical shell, is a different matter.
- The **additional** insulation **increases the conduction resistance** of the insulation layer but it also **decreases the convection resistance** of the surface **because** of the **increase** in the **outer surface area** for convection.
- Therefore, the **heat transfer** from a **pipe may increase or decrease**, depending on which effect dominates.

The Critical Radius of Insulation



The Critical Radius of Insulation

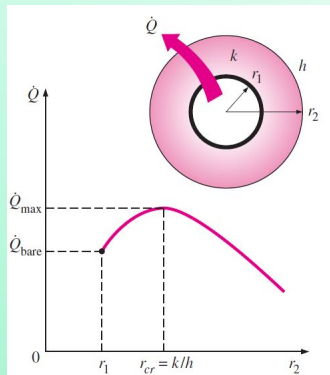


The rate of heat transfer from the insulated pipe to the surrounding air can be expressed as:

$$q_r = \frac{(T_1 - T_\infty)}{\frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi Lk} + \frac{1}{h(2\pi r_2 L)}}$$

The value of r_2 at which heat transfer rate reaches max. is determined from the requirement that $\frac{dq_r}{dr}$ (zero slope):

$$r_{cr,cylinder} = \frac{k}{h}$$



Problem: Critical Radius

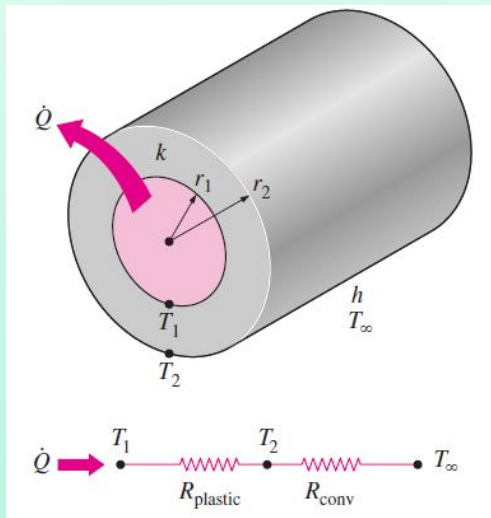


A 3 mm diameter and 6 m long electric wire is tightly wrapped with a 2 mm thick plastic cover whose thermal conductivity is $k = 0.15 \text{ W/m K}$. Electrical measurements indicate that a current of 10 A passes through the wire and there is a voltage drop of 8 V along the wire. If the insulated wire is exposed to a medium at 27°C with a heat transfer coefficient of $h = 12 \text{ W/m}^2 \text{ K}$, determine the temperature at the interface of the wire and the plastic cover in steady operation. Also determine whether doubling the thickness of the plastic cover will increase or decrease this interface temperature.

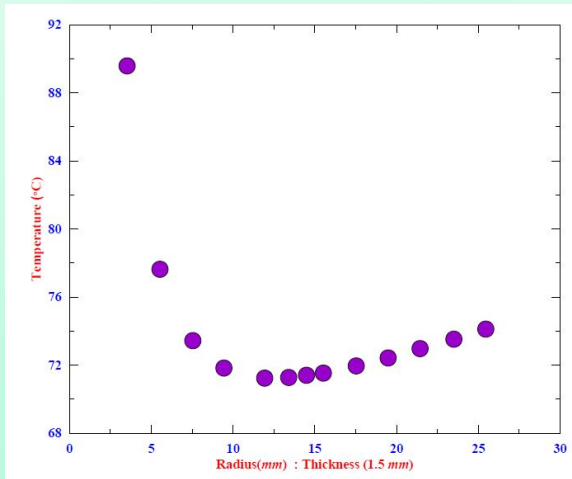
Ans: 89.5°C , 77.5°C

Hint: $q_r = VI$

Problem: Critical Radius



Problem: Critical Radius



	Plane Wall	Cylindrical Wall ^a	Spherical Wall ^a
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) = 0$	$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dT}{dr} \right) = 0$
Temperature distribution	$T_{s,1} - \Delta T \frac{x}{L}$	$T_{s,2} + \Delta T \frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T \left[\frac{1 - (r_1/r)}{1 - (r_1/r_2)} \right]$
Heat flux (q'')	$k \frac{\Delta T}{L}$	$\frac{k \Delta T}{r \ln(r_2/r_1)}$	$\frac{k \Delta T}{r^2 [(1/r_1) - (1/r_2)]}$
Heat rate (q)	$kA \frac{\Delta T}{L}$	$\frac{2\pi Lk \Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k \Delta T}{(1/r_1) - (1/r_2)}$
Thermal resistance ($R_{t,cond}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1) - (1/r_2)}{4\pi k}$

^aThe critical radius of insulation is $r_{cr} = k/h$ for the cylinder and $r_{cr} = 2k/h$ for the sphere.



A very common thermal energy generation process involves the conversion from electrical to thermal energy in a current carrying medium (resistance heating). The rate at which energy is generated by passing a current through a medium of electrical resistance R_e is:

$$\dot{E}_g = I^2 R_e$$

If this power generation occurs uniformly throughout the medium of volume V , the volumetric generation rate (W/m^3) is:

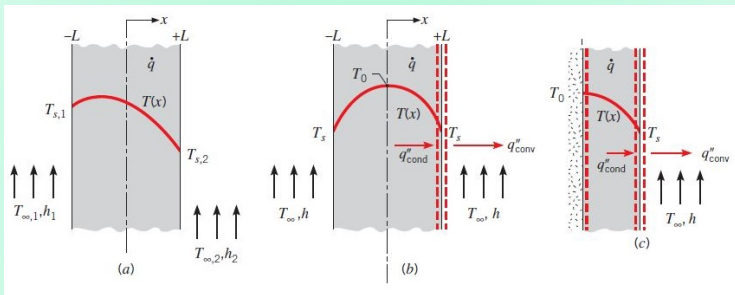
$$\dot{q} = \frac{\dot{E}_g}{V} = \frac{I^2 R_e}{V}$$

Conduction with \dot{E}_g in a Plane Wall



Consider a plane wall, in which there is uniform energy generation per unit volume (\dot{q} is constant) and the surfaces are maintained at $T_{s,1}$ and $T_{s,2}$. The appropriate form of the heat equation:

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \quad \left| \begin{array}{l} T(-L) = T_{s,1} \\ T(L) = T_{s,2} \end{array} \right.$$





$$T(x) = \frac{\dot{q}L^2}{2k} \left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2} \frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2}$$

The heat flux at any point in the wall may be determined by Fourier's Law. The **heat flux is not independent of x** .

Special case: $T_{s,1} = T_{s,2} = T_s$

The temperature distribution is then symmetrical about the central plane:

$$T(x) = \frac{\dot{q}L^2}{2k} \left(1 - \frac{x^2}{L^2}\right) + T_s$$

The maximum temperature exists at the central plane:

$$T(0) = T_0 = \frac{\dot{q}L^2}{2k} + T_s$$



$$\frac{T(x) - T_0}{T_s - T_0} = \left(\frac{x}{L}\right)^2$$

- It is important to note that the temperature gradient $\frac{dT}{dx} \Big|_{x=0} = 0$ at the plane of symmetry.
- No heat transfer across this plane - adiabatic surface.
- The above equation can also be applied to plane walls that are perfectly insulated on one side ($x = 0$) and a fixed T_s on the other side ($x = L$).

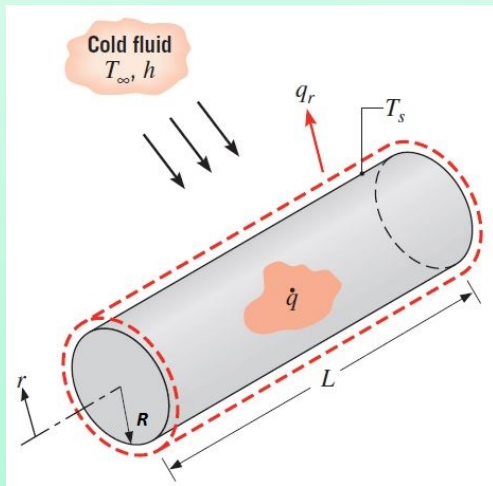
However, in most of the cases, T_s is an unknown. It is computed from the energy balance at the surface to the adjoining fluid:

$$\begin{aligned} -k \frac{dT}{dx} \Big|_{x=L} &= h(T_s - T_\infty) \\ \implies T_s &= T_\infty + \frac{\dot{q}L}{h} \end{aligned}$$

Conduction with \dot{E}_g in Radial Systems



Consider a long, solid cylinder (may be a current carrying wire).
For steady state conditions: $\dot{E}_g = q_{conv}$.





$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \frac{\dot{q}}{k} = 0$$

Boundary conditions:

$$\left. \frac{dT}{dr} \right|_{r=0} = 0, \quad T(R) = T_s \quad \text{and} \quad T(r=0) = T_0$$

$$\implies T(r) = T_s + \frac{\dot{q}R^2}{4k} \left(1 - \frac{r^2}{R^2} \right)$$

$$\text{or} \quad \frac{T(r) - T_s}{T_0 - T_s} = 1 - \left(\frac{r}{R} \right)^2$$

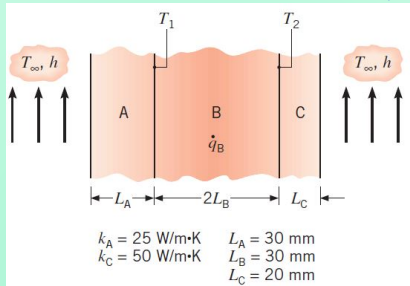
T_s can be obtained from the energy balance at the surface:

$$\dot{q} \pi R^2 L = h 2\pi R L (T_s - T_\infty)$$

$$T_s = T_\infty + \frac{\dot{q}R}{2h}$$

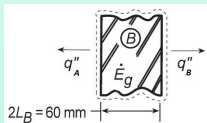
Consider one-dimensional conduction in a plane composite wall. The outer surfaces are exposed to a fluid at 25°C and a convection heat transfer coefficient of $1000\text{ W/m}^2\text{ K}$. The middle wall B experiences uniform heat generation \dot{q}_B , while there is no generation in walls A and C. The temperatures at the interfaces are $T_1 = 261^\circ\text{C}$ and $T_2 = 211^\circ\text{C}$. Assuming negligible contact resistance at the interfaces, determine \dot{q}_B and k_B .

Ans: $4 \times 10^6\text{ W/m}^3$, 15.3 W/m K .



From an energy balance on wall B,

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$



$$-q_A - q_C + \dot{q}_B V = 0$$

$$\Rightarrow -q''_A - q''_C + \dot{q}_B 2L_B = 0$$

$$\dot{q}_B = \frac{q_A + q_C}{2L_B}$$

Heat flow across ambient and wall A:

$$q''_A = \frac{T_1 - T_\infty}{\left(\frac{1}{h} + \frac{L_A}{k_A}\right)} = \frac{261 - 25}{\frac{1}{1000} + \frac{30 \times 10^{-3}}{25}} = 107272.7 \text{ W/m}^2$$



Heat flow across ambient and wall C:

$$q_C'' = \frac{T_1 - T_\infty}{\left(\frac{1}{h} + \frac{L_C}{k_C}\right)} = \frac{211 - 25}{\frac{1}{1000} + \frac{20 \times 10^{-3}}{50}} = 132857.1 \text{ W/m}^2$$

$$\dot{q}_B = \frac{q_A + q_C}{2L_B} = \frac{107272.7 + 132857.1}{60 \times 10^{-3}} = 4 \times 10^6 \text{ W/m}^3$$

$$T(x) = \frac{\dot{q}_B L_B^2}{2k_B} \left(1 - \frac{x^2}{L_B^2} \right) + \frac{T_2 - T_1}{2} \frac{x}{L_B} + \frac{T_1 + T_2}{2}$$

$$q_B''(x) = -k_B \left[\frac{\dot{q}_B}{2k_B} (-2x) + \frac{T_2 - T_1}{2L_B} \right]$$

$$q_B''|_{x=-L_B} = -\dot{q}_B L_B - \frac{T_2 - T_1}{2L_B} k_B$$

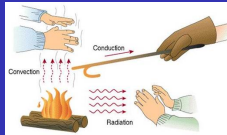
$$k_B = \frac{-q_A'' + \dot{q}_B L_B}{(T_1 - T_2)/2L_B} = 15.35 \text{ W/m K}$$



A plane wall is a composite of two materials, A and B. The wall of material A has uniform heat generation $\dot{q} = 1.5 \times 10^6 \text{ W/m}^3$, $k_A = 75 \text{ W/m K}$, and thickness $L_A = 50 \text{ mm}$. The wall material B has no generation with $k_B = 150 \text{ W/m K}$, and thickness $L_B = 20 \text{ mm}$. The inner surface of material A is well insulated, while the outer surface of material B is cooled by a water stream with $T_\infty = 30^\circ\text{C}$ and $h = 1000 \text{ W/m}^2 \text{ K}$.

- Sketch the temperature distribution that exists in the composite under steady state conditions.
- Determine the temperature T_0 of the insulated surface and the temperature T_2 of the cooled surface.

Heat and Mass Transfer



1D Heat Transfer from Extended Surfaces - Fins

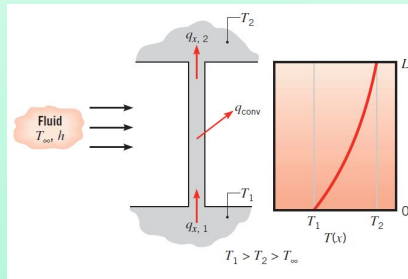
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Extended surface: solid that experiences energy transfer by conduction within its boundaries, as well as energy transfer by convection and/or radiation between its boundaries and the surround-



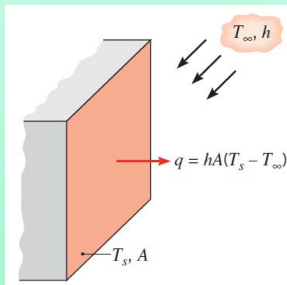
ings.

strut is used to provide mechanical support to two walls at different T . A temperature gradient in the x -direction sustains heat transfer by conduction internally, at the same time there is energy transfer by convection from the surface.

The most frequent application is one in which an **extended surface** is used specifically to **enhance** the heat transfer rate between a **solid and an adjoining fluid** - called as **fin**

Consider a plane wall:

$$q_{conv} = hA(T_s - T_\infty)$$



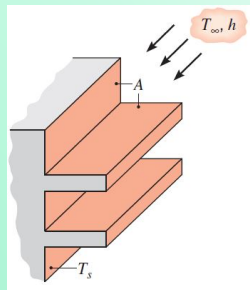
For fixed T_s , 2 ways to enhance the rate of heat transfer:

- Increase the fluid velocity: **cost** of blower or pump power
- T_∞ could be reduced: **impractical**

Limitations: Many situations would be encountered in which **increasing h** to the max. possible value is either **insufficient** to obtain the desired heat transfer rate or the associated **costs** are prohibitively **high**.

How about **increasing surface area for convection?**

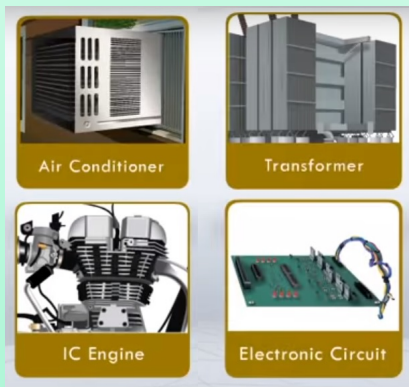
By providing **fins** that extend from the wall into the surrounding fluid.



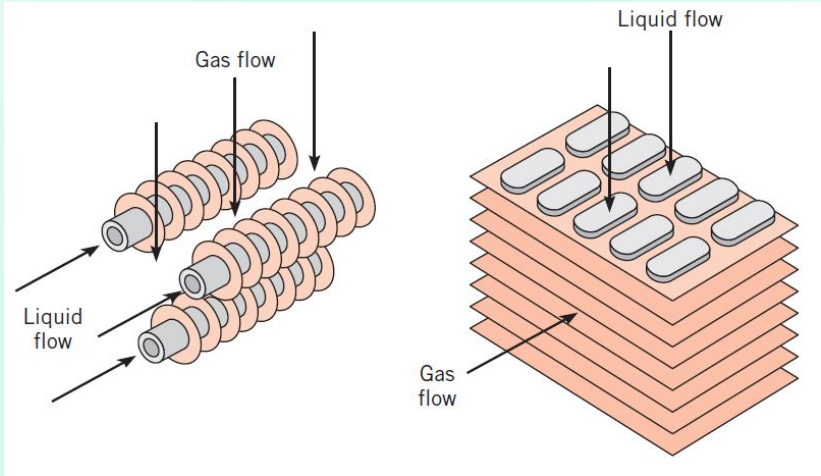


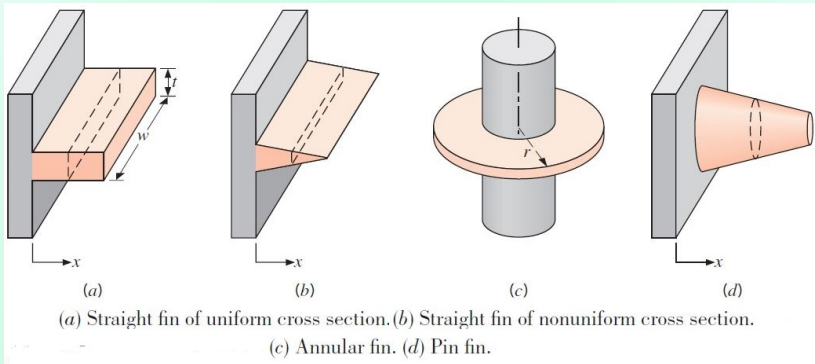
- k of the fin material has a strong effect on the temperature distribution along the fin and therefore influences the degree to which the heat transfer rate is enhanced.
- Ideally, the fin material should have a large k to minimize temperature variations from its base to its tip.
- In the limit of infinite thermal conductivity, the entire fin would be at the temperature of the base surface, thereby providing the maximum possible heat transfer enhancement.

- The arrangement for **cooling engine heads** on motorcycles and lawn-mowers
- For **cooling** electric power **transformers**
- The tubes with attached fins used to **promote heat exchange** between air and the working fluid of an **air conditioner**



Typical Finned Tube Heat Exchangers



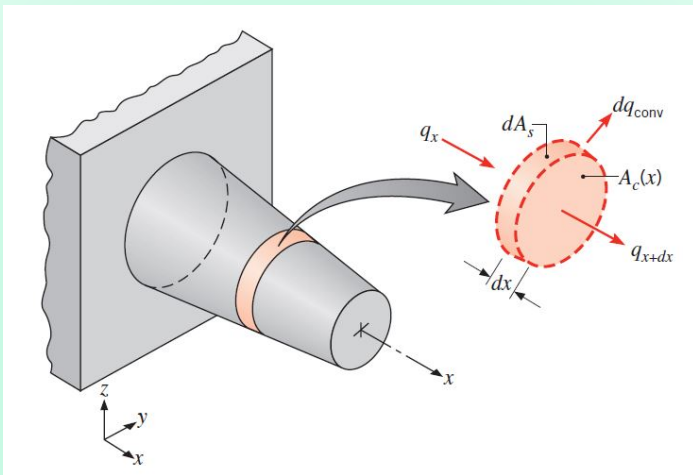


For an extended surface, the direction of heat transfer from the boundaries is **perpendicular to the principal direction** of heat transfer in the solid.

General Conduction Analysis



To determine the heat transfer rate associated with a fin, we must first obtain the temperature distribution along the fin.





- 1-D heat transfer (longitudinal x direction). In practice the fin is thin and the temperature changes in the longitudinal direction are much larger than those in the transverse direction.
- Steady state
- k is constant
- No heat generation
- Negligible radiation from the surface
- The rate at which the energy is convected to the fluid from any point on the fin surface must be balanced by the rate at which the energy reaches that point due to conduction in the transverse (y, z) direction.
- h is uniform over the surface

$$q_x = q_{x+dx} + dq_{conv}$$

However,

$$q_x = -kA_c \frac{dT}{dx}$$

A_c may vary with x .

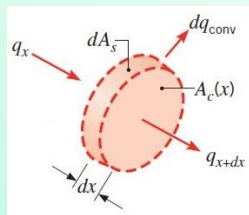
$$q_{x+dx} = q_x + \frac{dq_x}{dx} dx$$

$$q_{conv} = h dA_s (T - T_\infty)$$

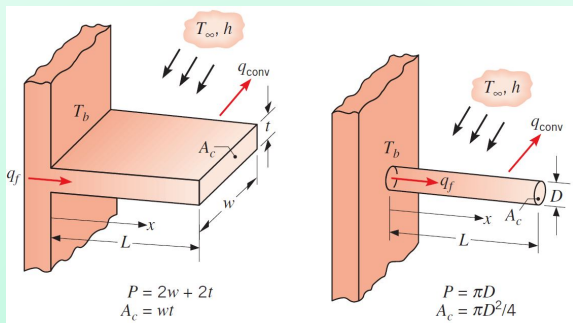
dA_s is the surface area of dx

$$\Rightarrow k \frac{d}{dx} \left(A_c \frac{dT}{dx} \right) dx - h dA_s (T - T_\infty)$$

$$\frac{d^2 T}{dx^2} + \left(\frac{1}{A_c} \frac{dA_c}{dx} \right) \frac{dT}{dx} - \left(\frac{1}{A_c} \frac{h dA_s}{k dx} \right) (T - T_\infty)$$



Fins of Uniform Cross-Sectional Area



- $T(0) = T_b$
- A_c is constant, $dA_c/dx = 0$
- $A_s = Px$ where x is measured from base, P is fin perimeter
- $dA_s/dx = P$

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0$$



Excess temperature, θ

$$\theta(x) = T(x) - T_{\infty}$$

$$d\theta/dx = dT/dx$$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

$$\text{where } m^2 = \frac{hP}{kA_c}$$

The above equation is a linear, homogeneous, second-order differential equation with constant coefficients. The general solution is of the form:

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

It is necessary to specify appropriate BCs for C_1 and C_2 .



One such condition may be specified in terms of the temperature at the base of the fin ($x = 0$):

$$\theta(0) = T_b - T_\infty = \theta_b$$

The second condition, specified at the fin tip ($x = L$), may correspond to any one of the four different physical conditions:

- A. h at the fin tip
- B. Adiabatic condition at the fin tip
- C. Prescribed temperature maintained at the fin tip
- D. Infinite fin (very long fin)



- A. Infinite fin (very long fin): As $L \rightarrow \infty, \theta_L \rightarrow 0$
- B. Adiabatic condition at the fin tip

$$\left. \frac{d\theta}{dx} \right|_{x=L} = 0$$

- C. h at the fin tip

$$hA_c[T(L) - T_\infty] = -kA_c \left. \frac{dT}{dx} \right|_{x=L} \implies h\theta(L) = -k \left. \frac{d\theta}{dx} \right|_{x=L}$$

- D. Prescribed temperature maintained at the fin tip: $\theta(L) = \theta_L$

Case A: Infinite fin (very long fin)



As $L \rightarrow \infty$, $\theta_L \rightarrow 0$ and $e^{-mL} \rightarrow 0$

$$\begin{aligned}\theta|_{x=0} &= \theta_b; & \theta|_{x=L} &= 0 \\ \theta_b &= C_1 e^{mx} + C_2 e^{-mx}; & C_1 e^{mL} + C_2 e^{-mL} &= 0\end{aligned}$$

Equation for infinite fin

$$\frac{\theta}{\theta_b} = e^{-mx}$$

$$q_f = -kA_c \left. \frac{d\theta}{dx} \right|_{x=0} = \sqrt{hPkA_c} \theta_b$$

Case B: Adiabatic Condition at the Fin Tip



$$\begin{aligned}\theta|_{x=0} &= \theta_b; & \left. \frac{d\theta}{dx} \right|_{x=L} &= 0 \\ \theta_b &= C_1 e^{mx} + C_2 e^{-mx}; & C_1 e^{mx} - C_2 e^{-mx} &= 0 \\ \frac{\theta}{\theta_b} &= \frac{e^{mx}}{1 + e^{2mL}} + \frac{e^{-mx}}{1 + e^{-2mL}}\end{aligned}$$

Equation for adiabatic condition

$$\frac{\theta}{\theta_b} = \frac{\cosh[m(L-x)]}{\cosh mL}$$

$$\text{Note: } \cosh A = \frac{e^A + e^{-A}}{2}$$



$$\begin{aligned}q_f &= -kA_c \left. \frac{dT}{dx} \right|_{x=0} = -kA_c \left. \frac{d\theta}{dx} \right|_{x=0} \\&= -kA_c m\theta_b \left[\frac{1}{1 + e^{2mL}} - \frac{1}{1 + e^{-2mL}} \right] \\&= \sqrt{hPkA_c}\theta_b \left[\frac{e^{mL} - e^{-mL}}{e^{mL} + e^{-mL}} \right]\end{aligned}$$

Rate of heat transfer: Adiabatic Condition

$$q_f = \sqrt{hPkA_c}\theta_b \tanh mL$$

Case C: h from the Fin Tip



A practical way to account for the heat loss from the fin tip is to replace the *fin length* L in the relation for the adiabatic tip case by a corrected length.

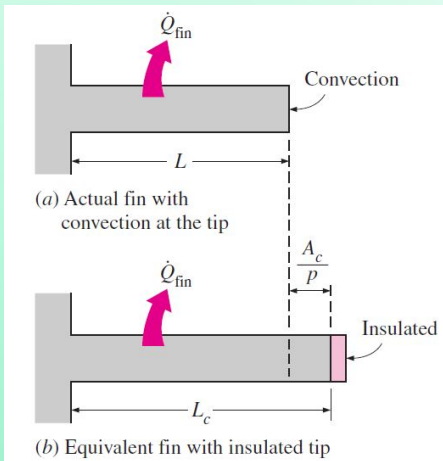
$$L_c = L + \frac{A_c}{P}$$

$$\frac{\theta}{\theta_b} = \frac{\cosh[m(L_c - x)]}{\cosh mL_c}$$

$$q_f = \sqrt{hPkA_c}\theta_b \tanh mL_c$$

$$L_{c, \text{rectangular fin}} = L + \frac{t}{2}$$

$$L_{c, \text{cylindrical fin}} = L + \frac{D}{4}$$



Uniform Cross-Sectional Fin: Summary



Temperature distribution & heat loss for fins of uniform cross-section

Tip Cond.	at $x = L$	$\frac{\theta}{\theta_b}$	q_f
Infinite fin	$\theta(L) = 0$	e^{-mx}	M
Adiabatic	$\left. \frac{d\theta}{dx} \right _{x=L} = 0$	$\frac{\cosh[m(L-x)]}{\cosh mL}$	$M \tanh mL$
Convection	$h\theta_L = -k \left. \frac{d\theta}{dx} \right _{x=L_c}$	$\frac{\cosh[m(L_c-x)]}{\cosh mL_c}$	$M \tanh mL$

$$m = \sqrt{\frac{hP}{kA_c}}; \quad M = \sqrt{hPkA_c}\theta_b; \quad L_c = L + \frac{A_c}{P}$$

A very long rod 5 mm in diameter has one end maintained at 100°C . The surface of the rod is exposed to ambient air at 25°C with a convection heat transfer coefficient of $100\text{ W/m}^2\text{ K}$.

- Determine the temperature distributions along rods constructed from pure copper, 2024 aluminium alloy and type AISI 316 stainless steel. What are the corresponding heat losses from the rods? Ans: 8.3 W, 5.6 W and 1.6 W
- Estimate how long the rods must be for the assumption of infinite length to yield an accurate estimate of the heat loss.

At $T = (T_b + T_{\infty})/2 = 62.5^{\circ}\text{C} = 335\text{ K}$:

$$k_{\text{copper}} = 398\text{ W/m K}$$

$$k_{\text{aluminium}} = 180\text{ W/m K}$$

$$k_{\text{stainless steel}} = 14\text{ W/m K}$$

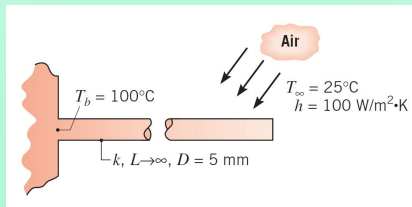
Hint: For an infinitely long fin: $\theta/\theta_b = e^{-mx}$

Find

- $T(x)$ and heat loss when rod is Cu, Al, SS.
- How long rods must be to assume infinite length.

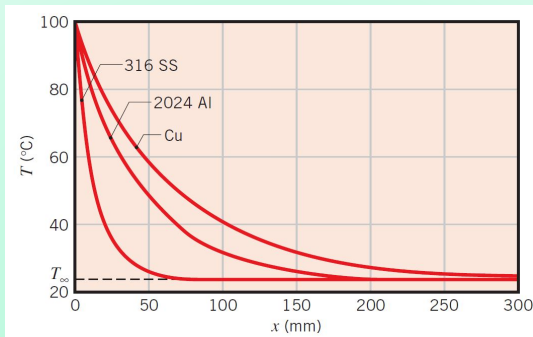
Assumptions

- Steady state conditions, 1-D along the rod
- Constant properties and uniform h
- Negligible radiation exchange with surroundings





$$T = T_{\infty} + (T_b - T_{\infty})e^{-m.x}$$



There

is little additional heat transfer associated with lengths more than 50 mm (SS), 200 mm (Al), and 300 mm (Cu).

$$q_f = \sqrt{hPkA_c}\theta_b$$



Since there is no heat loss from the tip of an infinitely long rod, an estimate of the validity of the approximation may be made by comparing q_f for infinitely long fin and adiabatic fin tip.

$$\sqrt{hPkA_c}\theta_b = \sqrt{hPkA_c}\theta_b \tanh mL$$

$$\tanh 4 = 0.999 \quad \text{and} \quad \tanh 2.5 = 0.987$$

$$\implies mL \geq 2.5$$

$$L \geq \frac{2.65}{m} = 2.5 \sqrt{\frac{kA_c}{hP}}$$

$$L_{Cu} = 0.18 \text{ m}; \quad L_{Al} = 0.12 \text{ m}; \quad \text{and} \quad L_{SS} = 0.033 \text{ m}$$



Comments

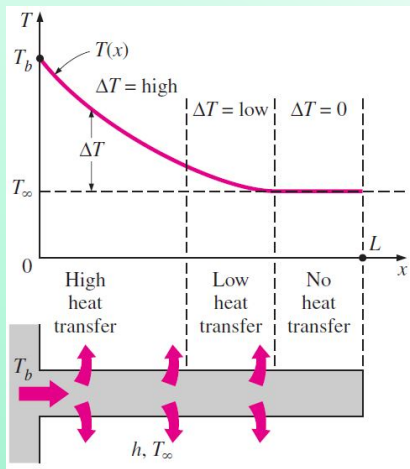
- The above results suggest that the fin heat transfer rate may accurately be predicted from the infinite fin approximation if $mL \geq 2.5$
- For more accuracy, if $mL \geq 4.6$:
 $L_{\infty} = 0.33 \text{ m (Cu)}, 0.23 \text{ m (Al)} \text{ and } 0.07 \text{ m (SS)}$

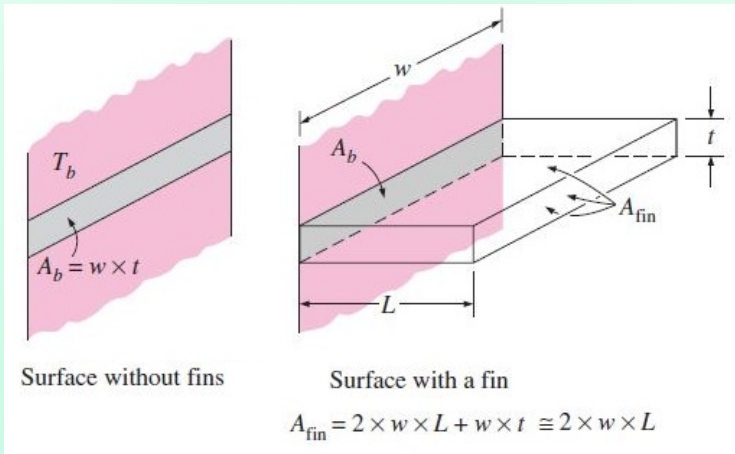
Proper Length of a Fin



$$\frac{q_{fin}}{q_{long\ fin}} = \tanh mL$$

mL	$\tanh mL$
0.1	0.100
0.2	0.197
0.5	0.462
1.0	0.762
1.5	0.905
2.0	0.964
2.5	0.987
3.0	0.995
4.0	0.999
5.0	1.000





No fin: $q_{\text{conv}} = hA_b(T_b - T_\infty)$



- The temperature of the fin will be T_b at the fin base and gradually decrease towards the fin tip.
- Convection from the fin surface causes the temperature at any cross-section to drop somewhat from the midsection toward the outer surfaces.
- However, the cross-sectional area of the fins is usually very small, and thus the temperature at any cross-section can be considered to be uniform.
- Also, the fin tip can be assumed for convenience and simplicity to be adiabatic by using the corrected length for the fin instead of the actual length.

In the limiting case of zero thermal resistance or infinite k , the temperature of fin will be uniform at the value of T_b . The heat transfer from the fin will be maximum in this case ($k \rightarrow \infty$):

$$q_{fin,max} = hA_{fin}(T_b - T_{\infty})$$

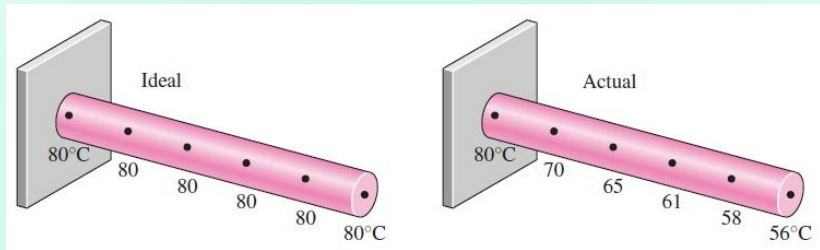


In reality, however, the temperature of the fin will drop along the fin and thus the heat transfer from the fin will be less because of the decreasing $[T(x) - T_\infty]$ toward the fin tip.

To account for the effect of this decrease in temperature on heat transfer, we define fin efficiency as:

$$\eta_{fin} = \frac{q_{fin}}{q_{fin,max}} = \frac{\text{Actual heat transfer rate from the fin}}{\text{Ideal heat transfer rate from the fin if the entire fin were at base temperature}}$$

Fin Efficiency





Case A: Infinitely long fins

$$\eta_{\text{long fin}} = \frac{q_{\text{fin}}}{q_{\text{fin,max}}} = \frac{1}{mL}$$

$$\therefore A_{\text{fin}} = pL$$

Case B: Adiabatic tip

$$\eta_{\text{adiabatic}} = \frac{q_{\text{fin}}}{q_{\text{fin,max}}} = \frac{\tanh mL}{mL}$$

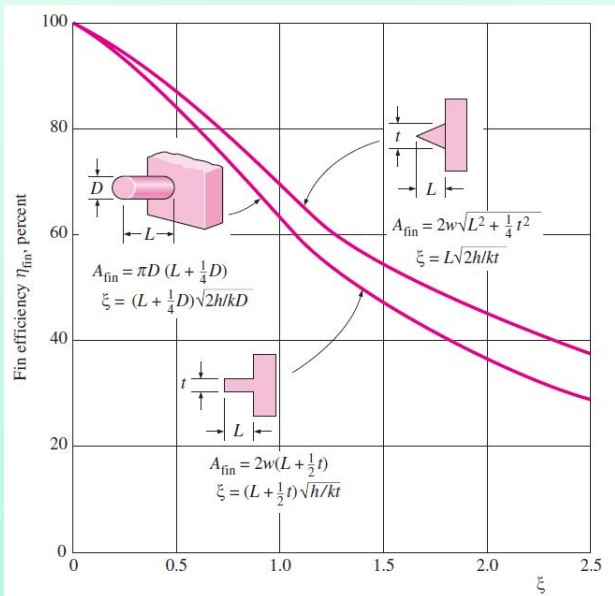
Case C: Convection at tip

$$\eta_{h \text{ at tip}} = \frac{q_{\text{fin}}}{q_{\text{fin,max}}} = \frac{\tanh mL_c}{mL_c}$$

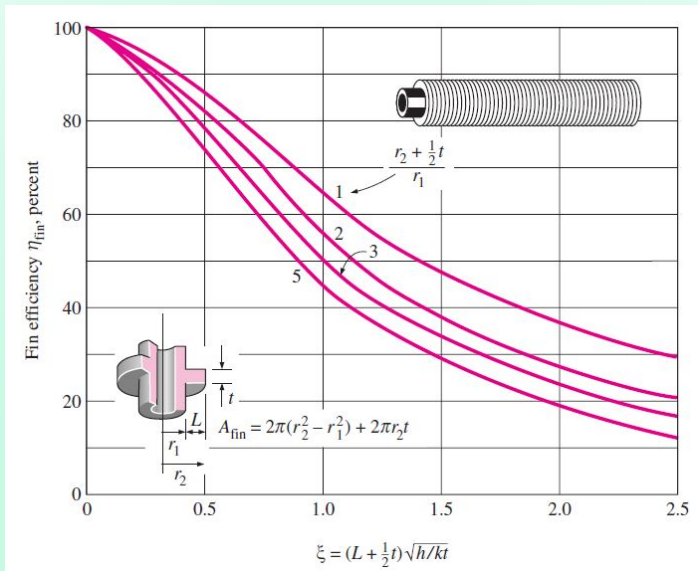


- An important consideration in the design of finned surface is the selection of the **proper fin length, L** .
- Normally the **longer the fin**, the larger the heat transfer and thus the **higher the rate of heat transfer** from the fin.
- But **also** the larger the fin, the bigger the mass, the **higher the price**, and the **larger the fluid friction**.
- Therefore, **increasing** the length of the fin **beyond** a **certain value cannot be justified** unless the added benefits outweigh the added cost.
- Also, η_{fin} decreases with **increasing fin length** because of the decrease in fin temperature with length.
- Fin lengths that cause the fin efficiency to drop below 60% usually cannot be justified economically and should be avoided.
- η of most fins used in practice is **$> 90\%$** .

η of Rectangular, Triangular, Parabolic Profiles



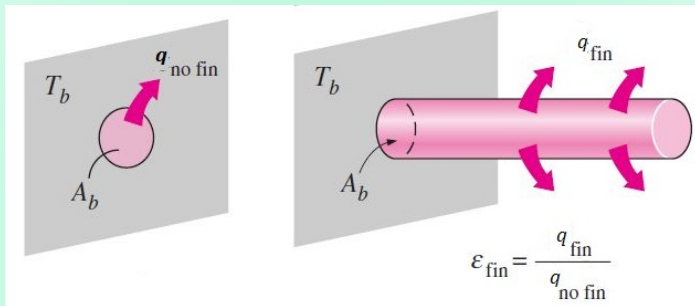
η of Annular fins of constant thickness, t



The performance of the fins is judged on the basis of enhancement of heat transfer relative to the no fin case.

$$\epsilon_{fin} = \frac{q_{fin}}{q_{no\ fin}} = \frac{q_f}{hA_b(T_b - T_\infty)}$$

where A_b is the fin cross-sectional area at the base.





- $\epsilon_{fin} = 1$ indicates that the addition of fins to the surface does not affect heat transfer at all. That is, heat conducted to the fin through the base area A_b is equal to the heat transferred from the same area A_b to the surrounding medium.
- $\epsilon_{fin} < 1$ indicates that the fin actually acts as insulation, slowing down the heat transfer from the surface. This situation can occur when fins made of low k are used.
- $\epsilon_{fin} > 1$ indicates that the fins are enhancing heat transfer from the surface. However, the use of fins cannot be justified unless ϵ_{fin} is sufficiently larger than 1 (≥ 2). Finned surfaces are designed on the basis of maximizing effectiveness of a specified cost or minimizing cost for a desired effectiveness.



η_{fin} and ε_{fin} are related to performance of the fin, but they are different quantities.

$$\begin{aligned}\varepsilon_{fin} &= \frac{q_{fin}}{q_{no\ fin}} \\ &= \frac{q_{fin}}{hA_b(T_b - T_\infty)} \\ &= \frac{\eta_{fin}hA_{fin}(T_b - T_\infty)}{hA_b(T_b - T_\infty)} \\ \Rightarrow \quad \varepsilon_{fin} &= \frac{\eta_{fin}A_{fin}}{A_b}\end{aligned}$$

Therefore, η_{fin} can be determined easily when ε_{fin} is known, or vice versa.



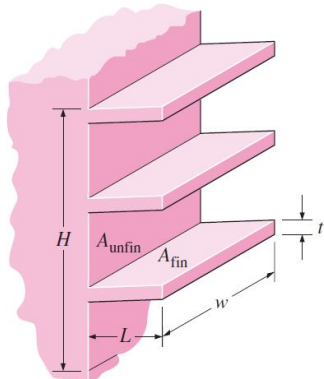
$$\varepsilon_{fin} = \frac{q_{fin}}{q_{no\ fin}} = \frac{\sqrt{hPkA_c}\theta_b}{hA_b(T_b - T_\infty)} = \sqrt{\frac{kP}{hA_c}}$$

$\because A_c = A_b$ and $\theta_b = T_b - T_\infty$

- k of fin should be high. Ex: **Cu, Al, Fe**. Aluminium is low cost, weight, and resistant to corrosion.
- P/A_c should be high. **Thin plates or slender pin fins**
- h should be low. **Gas** instead of liquid; **Natural convection** instead of forced convection. Therefore, in liquid-to-gas heat exchangers (**car radiators**), fins are placed on the **gas side**.

Heat transfer rate for a surface containing n fins:

$$\begin{aligned}q_{tot,fin} &= q_{unfin} + q_{fin} \\ &= hA_{unfin}(T_b - T_\infty) \\ &\quad + \eta_{fin}A_{fin}(T_b - T_\infty)\end{aligned}$$



$$\begin{aligned}A_{no\ fin} &= w \times H \\ A_{unfin} &= w \times H - 3 \times (t \times w) \\ A_{fin} &= 2 \times L \times w + t \times w \text{ (one fin)} \\ &\approx 2 \times L \times w\end{aligned}$$

$$q_{tot,fin} = h(A_{unfin} + \eta_{fin}A_{fin})(T_b - T_\infty)$$



We can also define an overall effectiveness for a finned surface as the ratio of the total q from the finned surface to the q from the same surface if there were no fins:

$$\begin{aligned}\varepsilon_{fin,overall} &= \frac{q_{fin}}{q_{nofin}} \\ &= \frac{h(A_{unfin} + \eta_{fin}A_{fin})(T_b - T_\infty)}{hA_{nofin}(T_b - T_\infty)}\end{aligned}$$

A_{nofin} is the area of the surface when there are no fins

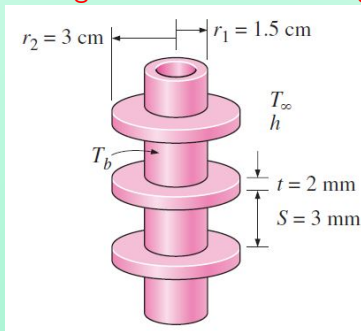
A_{fin} is the total surface area of all the fins on the surface

A_{unfin} is the area of the unfinned portion of the surface.

$\varepsilon_{fin,overall}$ depends on number of fins per unit length as well as ε_{fin} of individual fins.

$\varepsilon_{fin,overall}$ is a better measure of the performance than ε_{fin} of individual fins.

Steam in a heating system flows through tubes: outer diameter is $D_1 = 3$ cm and whose walls are maintained at 125°C . Circular aluminium fins ($k = 180$ W/m K) of $D_2 = 6$ cm, $t = 2$ mm are attached. The space between the fins is 3 mm, and thus there are 200 fins per meter length of the tube. Surrounded air: $T_\infty = 27^\circ\text{C}$, $h = 60$ W/m² K. Determine the increase in heat transfer from the tube per meter of its length as a result of adding fins.





Known

Properties of the fin, ambient conditions, heat transfer coefficient, dimensions of the fin.

Find

Increase in heat transfer from the tube per meter of its length as a result of adding fins.

Assumptions

- Steady state conditions, 1-D along the rod
- Constant properties and uniform h
- Negligible radiation exchange with surroundings

Solution: Analysis



In case of no fins (per unit length, $l = 1$ m:

$$A_{nofin} = \pi D_1 l = 0.0942 \text{ m}^2$$

$$q_{nofin} = h A_{nofin} (T_b - T_\infty) = 554 \text{ W}$$

$$r_1 = D_1/2 = 0.015 \text{ m}$$

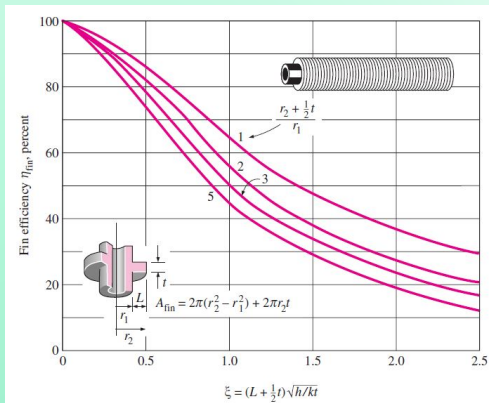
$$r_2 = D_2/2 = 0.03 \text{ m}$$

$$\frac{r_2 + \frac{t}{2}}{r_1} = \mathbf{2.07}$$

$$L = r_2 - r_1 = 0.015 \text{ m}$$

$$\xi = \left(L + \frac{t}{2} \right) \sqrt{\frac{h}{kt}} = \mathbf{0.207}$$

$$\Rightarrow \boxed{\eta_{fin} = 0.95}$$



q from finned portion

$$\begin{aligned}A_{fin} &= 2\pi (r_2^2 - r_1^2) + 2\pi r_2 t \\ &= 0.00462 \text{ m}^2\end{aligned}$$

$$\begin{aligned}q_{fin} &= \eta_{fin} q_{fin,max} \\ &= \eta_{fin} h A_{fin} (T_b - T_\infty) \\ &= 27.81 \text{ W}\end{aligned}$$

q from unfinned portion of tube

$$\begin{aligned}A_{unfin} &= 2\pi r_1 S \\ &= 0.000283 \text{ m}^2 \\ q_{unfin} &= h A_{unfin} (T_b - T_\infty) \\ &= 1.67 \text{ W}\end{aligned}$$

There are 200 fins per meter length of the tube. The total heat transfer from the finned tube:

$$q_{tot,fin} = n(q_{fin} + q_{unfin}) = 5896 \text{ W}$$

\therefore the increase in heat transfer from the tube per meter of its length as a result of the addition of fins is:

$$q_{increase} = q_{tot,fin} - q_{nofin} = \mathbf{5342 \text{ W}} \quad \text{per meter tube length}$$



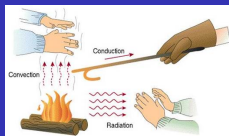
Effectiveness

The overall effectiveness of the finned tube is:

$$\varepsilon_{fin,overall} = \frac{Q_{tot,fin}}{Q_{tot,nofin}} = 10.6$$

That is, the rate of heat transfer from the steam tube increases by a factor of 10 as a result of adding fins.

Heat and Mass Transfer



Transient Heat Conduction

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Time dependent conduction - Temperature history inside a conducting body that is immersed suddenly in a bath of fluid at a different temperature.

Ex: Quenching of special alloys, heat treatment of bearings

The temperature of such a body varies with time as well as position.

$$T(x, y, z, t)$$

Transient Heat Conduction



A body is exposed to ambient

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

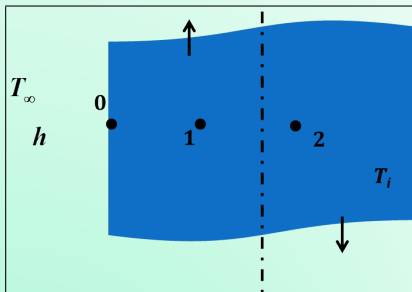
No heat generation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$\alpha \rightarrow$ Thermal diffusivity (m^2/s)

It appears only in the transient conduction

$$T = f(x, t)$$



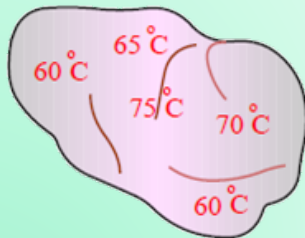
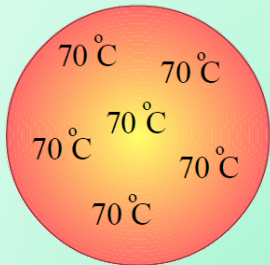
$$T_{t=0} = T_i$$

$$\left. \frac{\partial T}{\partial x} \right|_{x=0} = 0$$

$$q_{x=\pm L} = h(T_\infty - T)$$

Lumped: Temperature is essentially uniform throughout the body.

$$T(x, y, z, t) = T(t)$$

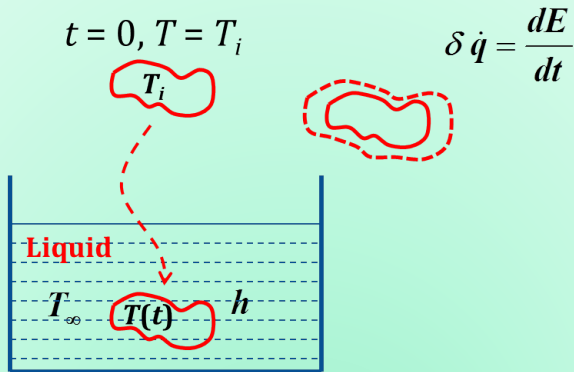


Copper ball with uniform temperature Pot

Lumped Capacitance Model



Hot forging that is initially at uniform temperature, T_i and is quenched by immersing it in a liquid of lower temperature $T_\infty < T_i$



$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen} = \dot{E}_{st}$$



$$-hA(T - T_{\infty}) = \rho V C_p \frac{dT}{dt}$$

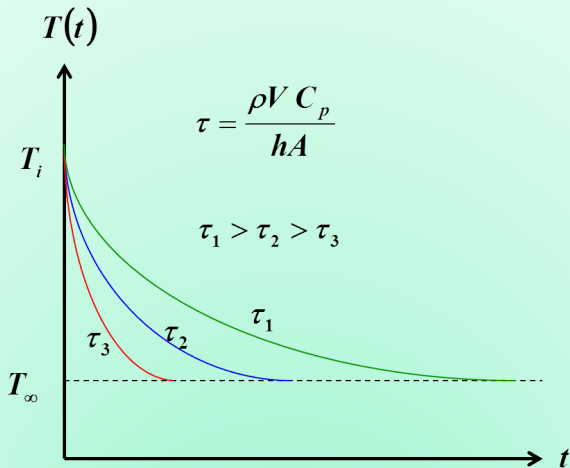
$$\int_{T=T_i}^T \frac{dT}{T - T_{\infty}} = -\frac{hA}{\rho V C_p} \int_{t=0}^t dt$$

$$\frac{\theta}{\theta_i} = \frac{T - T_{\infty}}{T_i - T_{\infty}} = e^{-\frac{t}{\tau}} \quad \tau = \frac{\rho V C_p}{hA}$$

$$\tau = \left(\frac{1}{hA} \right) (\rho V C_p) = R_t C_t$$

R_t - Resistance to convection heat transfer

C_t - Lumped thermal capacitance of the solid



$$\left. \frac{\theta}{\theta_i} \right|_{t=\tau} = 0.368$$



The rate of convection heat transfer between the body and its environment at any time: $q = hA[T(t) - T_\infty]$

Total energy transfer occurring up to sometime, t :

$$\begin{aligned} Q &= \int_{t=0}^t q dt \\ &= \int_{t=0}^t hA[T(t) - T_\infty] dt \\ &= hA(T_i - T_\infty) \int_{t=0}^t e^{-\frac{t}{\tau}} \end{aligned}$$

$$Q = \rho V C_p (T_i - T_\infty) \left[1 - e^{-\frac{t}{\tau}} \right]$$

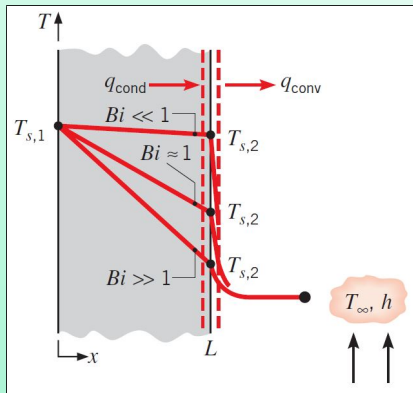
Consider a body exposed to ambient

$$q_{conv} = q_{cond}$$

$$h(T_{s,2} - T_{\infty}) = k \frac{T_{s,1} - T_{s,2}}{L_c}$$

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_{\infty}} = \frac{hL_c}{k} = Bi$$

$$L_c = \frac{V}{A}$$



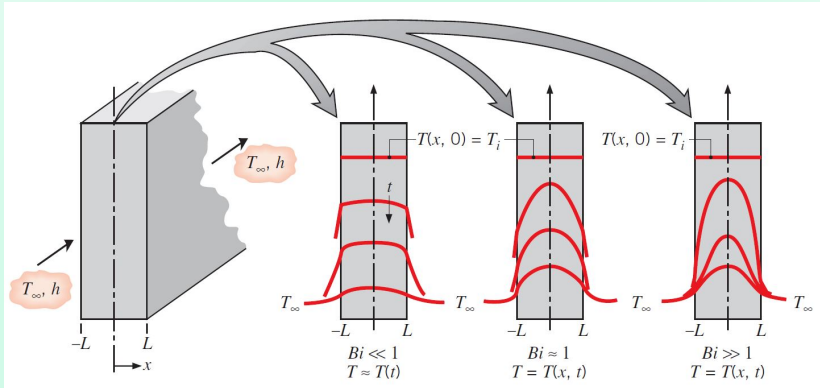


$$\begin{aligned} \text{Bi} &= \frac{hL_c}{k} \\ &= \frac{h\Delta T}{k\Delta T/L_c} \\ &= \frac{\text{Conv. at the surface of the body}}{\text{Conduction within the body}} \end{aligned}$$

$$\begin{aligned} \text{Bi} &= \frac{L_c/k}{1/h} \\ &= \frac{\text{Conduction resistance within the body}}{\text{Conv. resistance at the surface}} \end{aligned}$$

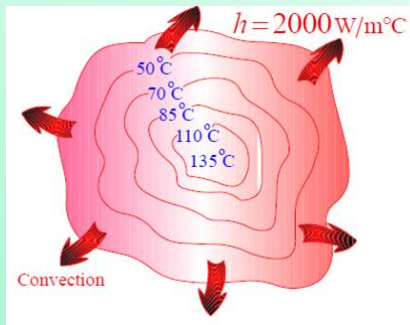
Generally accepted, $\boxed{\text{Bi} \leq 0.1}$ for assuming lumped.

Biot Number



Small bodies with **higher k** and **low h** are most likely satisfy $Bi \leq 0.1$.

When k is low and h is high, large temperature differences occur between the inner and outer regions of the body.





Jean-Baptiste Biot
(1774-1862)

- French physicist, astronomer, and mathematician born in Paris, France.
- Professor of mathematical physics at Collège de France .
- At the age of 29, he worked on the analysis of heat conduction even earlier than Fourier did (unsuccessful). After 7 years, Fourier read Biot's work.
- Awarded the Rumford Medal of the Royal Society in 1840 for his contribution in the field of Polarization of light.

Problem: Thermocouple Diameter

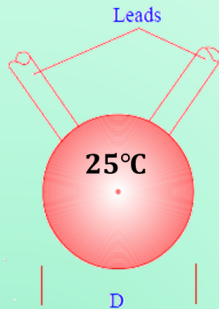


Determine the thermocouple junction diameter needed to have a time constant of one second.

Ambient: $T_{\infty} = 200^{\circ}\text{C}$, $h = 400 \text{ W/m}^2 \text{ K}$

Material

properties: $k = 20 \text{ W/m K}$, $C_p = 400 \text{ J/kg K}$, $\rho = 8500 \text{ kg/m}^3$ Ans:



0.706 mm



Known

Thermo-physical properties of the thermocouple junction used to measure the temperature of a gas stream.
Thermal environmental conditions.

Find

Junction diameter needed for a time constant of 1 second.

Assumptions

- Temperature of the junction is uniform at any instant.
- Radiation exchange with the surroundings is negligible.
- Losses by conduction through the leads is negligible.
- Constant properties.



$$L_c = \frac{V}{A} = \frac{\pi D^3/6}{\pi D^2} = \frac{D}{6}$$

$$\tau = \frac{\rho C_p V}{hA} = \frac{\rho C_p D}{6h}$$

$$D = 0.706 \text{ mm}$$

$$\text{Bi} = \frac{hL_c}{k} = 2.35 \times 10^{-3} < 0.1$$

Criterion for using the lumped capacitance model is satisfied and the lumped capacitance method may be used to an excellent approximation.

Comments

Heat transfer due to radiation exchange between the junction and the surroundings and conduction through the leads would affect the time response of the junction and would, in fact, yield an equilibrium temperature that differs from T_∞ .

Problem: Predict the Time of Death



A person is found dead at 5 PM in a room. The temperature of the body is measured to be 25°C when found. Estimate the time of death of that person.

$$T_{\infty} = 20^{\circ}\text{C}$$

$$h = 8 \text{ W/m}^2\cdot\text{K}$$



Known

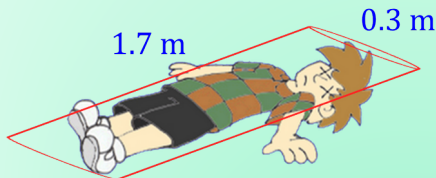
T of the person at 5 PM.
Thermal environmental conditions.

Find

The time of death of the person is to be estimated.

Assumptions

- The body can be modeled as a cylinder.
- Radiation exchange with the surroundings is negligible.
- The initial temperature of the person is 37°C .
- Assuming properties of water.



Water at $(37 + 25)/2 = 31^{\circ}\text{C}$

$k = 0.617 \text{ W/m.K}$

$C_p = 4.178 \text{ kJ/kg.K}$

$\rho = 996 \text{ kg/m}^3$

$$L_c = \frac{V}{A} = \frac{(\pi D^2/4)L}{\pi DL + 2(\pi D^2/4)} = 0.069 \text{ m}$$



$$\text{Bi} = \frac{hL_c}{k} = 0.9 > 0.1$$

Comment: Criterion for using the lumped capacitance model is not satisfied. However, let us get a rough estimate.

$$\tau = \frac{\rho C_p V}{hA} = \frac{\rho C_p V}{hA} = 35891 \text{ s}$$

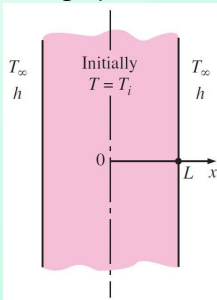
$$\frac{T - T_\infty}{T_i - T_\infty} = \frac{25 - 20}{37 - 20} = e^{-\frac{t}{\tau}}$$

$$t = 43923 \text{ s} = 12.2 \text{ hours}$$

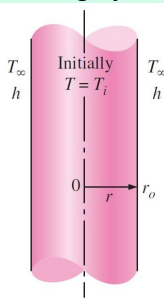
Therefore, the person would have died around 5 AM.



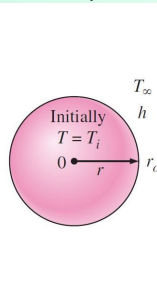
A large plane wall



A long cylinder



A sphere



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

$$T(x, 0) = T_i \quad \text{Initial}$$

$$\left. \frac{\partial T}{\partial x} \right|_{x=0} = 0 \quad \text{Symmetry}$$

$$-k \left. \frac{\partial T}{\partial x} \right|_{x=L} = h [T(L, t) - T_\infty] \quad \text{Boundary}$$

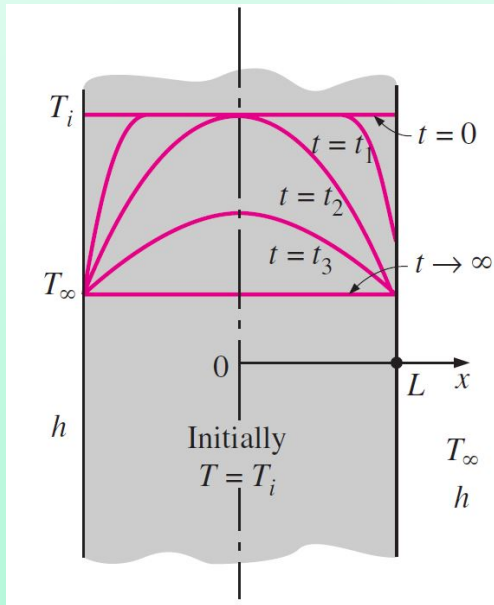


In all these three cases posses geometric and thermal symmetry:
the plane wall is symmetric about its center plane ($x = 0$),
the cylinder is symmetric about its center line($r = 0$), and
the sphere is symmetric about its center point ($r = 0$)

Neglect q_{rad} or incorporate as h_r .

The solution, however, involves infinite series, which are inconvenient and time consuming to evaluate.

Transient Temperature Profiles





It involves the parameters, $x, L, t, k, \alpha, h, T_i$, and T_∞ which are too many to make any graphical presentation of the results practical.

Dimensionless temperature: $\theta(x, t) = \frac{T(x, t) - T_\infty}{T_i - T_\infty}$

Dimensionless distance from center: $X = \frac{x}{L}$

Dimensionless h (Biot number): $Bi = \frac{hL}{k}$

Dimensionless time (Fourier number): $Fo = \frac{\alpha t}{L^2} = \tau$

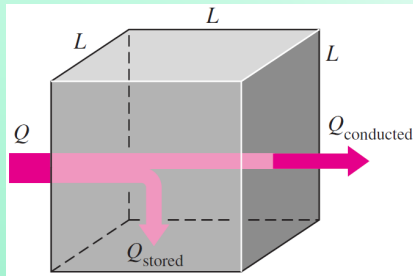
The non-dimensionalization enables us to present the temperature in terms of three parameters only: $\theta = f(X, Bi, Fo)$.

In case of lumped system analysis, $\theta = f(Bi, Fo)$.

$$Fo = \frac{\alpha t}{L^2} = \frac{(kL^2 \frac{\Delta T}{L})}{(\rho L^3 C_p \frac{\Delta T}{t})}$$

$$= \frac{\text{Rate of heat conducted across } L \text{ of a body of volume } L^3}{\text{Rate of heat stored in a body of volume } L^3}$$

$$Fo = \frac{Q_{\text{conducted}}}{Q_{\text{stored}}}$$





The exact solution involves infinite series. However, the terms in terms in the solutions converge rapidly with increasing time, and for $\tau > 0.2$, keeping the first term and neglecting all the remaining terms in the series results in an error under 2%.

$$\text{Plane Wall: } \theta(x, t)_{wall} = \frac{T(x, t) - T_{\infty}}{T_i - T_{\infty}} = A_1 e^{-\lambda_1^2 \tau} \cos(\lambda_1 x / L)$$

$$\text{Cylinder: } \theta(r, t)_{cyl} = \frac{T(r, t) - T_{\infty}}{T_i - T_{\infty}} = A_1 e^{-\lambda_1^2 \tau} J_0(\lambda_1 r / r_0)$$

$$\text{Sphere: } \theta(r, t)_{sph} = \frac{T(r, t) - T_{\infty}}{T_i - T_{\infty}} = A_1 e^{-\lambda_1^2 \tau} \frac{\sin(\lambda_1 r / r_0)}{\lambda_1 r / r_0}$$

A_1 and λ_1 are functions of the Bi number only.

Note: $\cos(0) = J_0(0) = 1$ and the limit of $(\sin x)/x = 1$.

$$\theta_0 = \frac{T_0 - T_{\infty}}{T_i - T_{\infty}} = A_1 e^{-\lambda_1^2 \tau}$$

Exact Solution: One Term Approximation

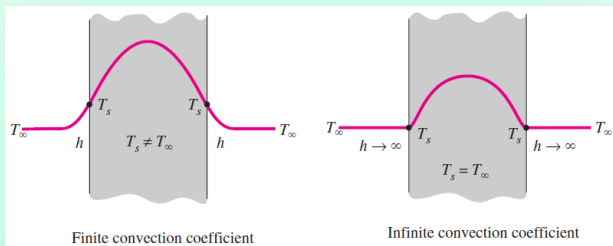


Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders, and spheres ($Bi = hL/k$ for a plane wall of thickness $2L$, and $Bi = hr_o/k$ for a cylinder or sphere of radius r_o)

Bi	Plane Wall		Cylinder		Sphere	
	λ_1	A_1	λ_1	A_1	λ_1	A_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

The zeroth- and first-order Bessel functions of the first kind

ξ	$J_0(\xi)$	$J_1(\xi)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	-0.0968	-0.4708
2.8	-0.1850	-0.4097
3.0	-0.2601	-0.3391
3.2	-0.3202	-0.2613



$$\frac{1}{\text{Bi}} = \frac{k}{hL} = 0 \text{ corresponds to } h \rightarrow \infty,$$

which corresponds to **specified surface temperature**, T_∞ , case.



The maximum amount of heat that a body can gain (or lose if $T_i > T_\infty$) is the change in the energy content of the body:

$$Q_{max} = mC_p(T_\infty - T_i) = \rho VC_p(T_\infty - T_i)$$

Q_{max} represents the amount of heat transfer for $t \rightarrow \infty$. The amount of heat transfer, Q at a finite time t is obviously less than this. It can be expressed as the sum of the internal energy changes throughout the entire geometry as:

$$Q = \int_V \rho C_p [T(x, t) - T_i] dV$$

Assuming constant properties:

$$\frac{Q}{Q_{max}} = \frac{\int_V \rho C_p [T(x, t) - T_i] dV}{\rho VC_p (T_\infty - T_i)} = \frac{1}{V} \int_V (1 - \theta) dV$$

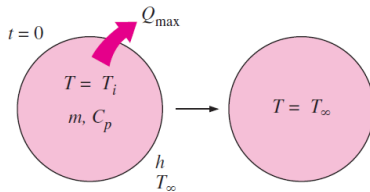


$$\text{Plane Wall: } \left(\frac{Q}{Q_{max}} \right)_{wall} = 1 - \theta_{0,wall} \frac{\sin \lambda_1}{\lambda_1}$$

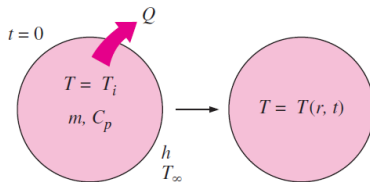
$$\text{Cylinder: } \left(\frac{Q}{Q_{max}} \right)_{cyl} = 1 - \theta_{0,cyl} \frac{J_1(\lambda_1)}{\lambda_1}$$

$$\text{Sphere: } \left(\frac{Q}{Q_{max}} \right)_{sph} = 1 - \theta_{0,wall} \frac{\sin \lambda_1 - \lambda_1 \cos \lambda_1}{\lambda_1}$$

Fraction of Total Heat Transfer: Gröber Chart



(a) Maximum heat transfer ($t \rightarrow \infty$)



$$\left. \begin{aligned} \text{Bi} &= \dots \\ \frac{h^2 \alpha t}{k^2} &= \text{Bi}^2 \tau = \dots \end{aligned} \right\} \frac{Q}{Q_{\max}} = \dots$$

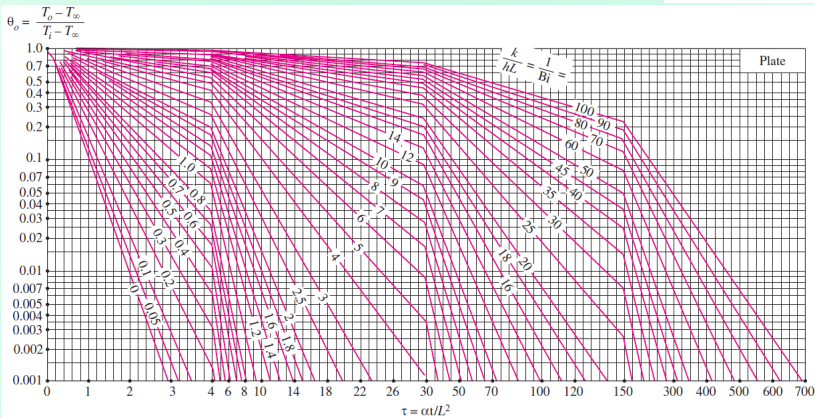
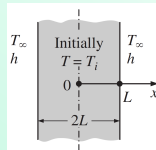
(Gröber chart)

Graphical Solution: Heisler Charts



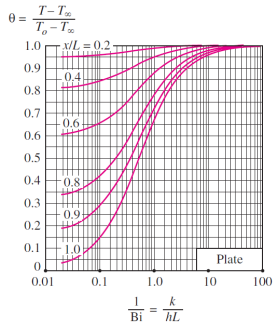
Valid for $\tau > 0.2$

Plane wall

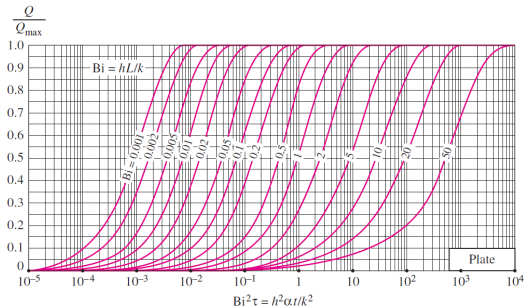


(a) Midplane temperature (from M. P. Heisler)

Graphical Solution: Heisler Charts



(b) Temperature distribution (from M. P. Heisler)



(c) Heat transfer (from H. Gröber et al.)



Limitations of One-term solution and Heisler/Gröber charts

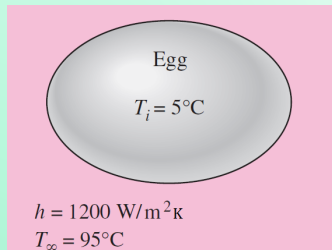
- Body is initially at a uniform temperature
- T_∞ and h are constant and uniform
- No energy generation within the body

Infinitely Large or Long?

- A plate whose thickness is small relative to the other dimensions can be modeled as an infinitely large plate, except very near the outer edges.
- The edge effects on large bodies are usually negligible.
- Ex: A large plane wall such as the wall of a house.
- Similarly, a long cylinder whose diameter is small relative to its length can be analyzed as an infinitely long cylinder.

An ordinary egg can be approximated as a 5 cm diameter sphere. The egg is initially at a uniform temperature of 5°C and is dropped into boiling water at 95°C . Taking the convection heat transfer coefficient to be $h = 1200 \text{ W/m}^2 \text{ K}$, determine how long it will take for the center of the egg to reach 70°C .

The water content of eggs is about 74%, and thus k and α of eggs can be approximated by those of water at the average temperature $(5 + 70)/2 = 37.5^{\circ}\text{C}$.



$$k = 0.627 \text{ W/m K}; \quad \alpha = k/\rho c_p = 0.151 \times 10^{-6} \text{ m}^2/\text{s}$$



$$T_i = 5^\circ\text{C}; \quad T_\infty = 95^\circ\text{C} \quad T_0 = 70^\circ\text{C} \quad h = 1200 \text{ W/m}^2 \text{ K}$$

$$k = 0.627 \text{ W/m K}; \quad \alpha = k/\rho c_p = 0.151 \times 10^{-6} \text{ m}^2/\text{s}$$

$$\text{Bi}_{L_c} = \frac{hL_c}{k} = 16 \quad \text{Bi}_{r_0} = \frac{hr_0}{k} = 47.8$$

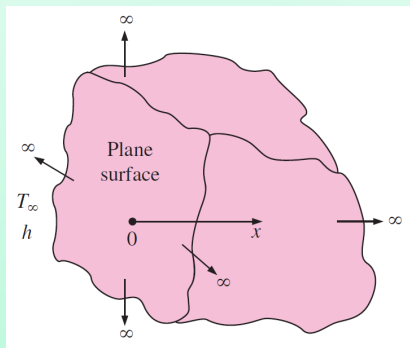
One-term approximation

$$\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}, \tau > 0.2$$

$$\tau = \frac{\alpha t}{r_0^2}$$

$$\lambda_1 = 3.0754; \quad A_1 = 1.9958$$

$$\tau = 0.209, \text{ Ans: } 14.4 \text{ min}$$



- A **single plane surface** and extends to infinity in all directions.
- Ex: Earth -temperature variation near its surface
Thick wall - temperature variation near one of its surfaces
- For short periods of time, most bodies are semi-infinite solids.
- The thickness of the body does not enter into the heat transfer analysis.



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

$$T(0, t) = T_s; \quad T(x \rightarrow \infty, t) = T_i$$

$$T(x, 0) = T_i$$

Convert PDE into ODE by combining the two independent variables x and t into a single variable η :

$$\eta = \frac{x}{\sqrt{4\alpha t}}$$

Similarity variable, η

- $\eta = 0$ at $x = 0$
- $\eta \rightarrow \infty$ at $x \rightarrow \infty$
- $\eta \rightarrow \infty$ at $t = 0$



$$\begin{aligned}\frac{\partial T}{\partial t} &= \frac{dT}{d\eta} \frac{d\eta}{dt} = -\frac{x}{2t\sqrt{4\alpha t}} \frac{dT}{d\eta} \\ \frac{\partial T}{\partial x} &= \frac{dT}{d\eta} \frac{d\eta}{dx} = \frac{1}{\sqrt{4\alpha t}} \frac{dT}{d\eta} \\ \frac{\partial^2 T}{\partial x^2} &= \frac{d}{d\eta} \left(\frac{\partial T}{\partial x} \right) \frac{d\eta}{dx} = \frac{1}{4\alpha t} \frac{d^2 T}{d\eta^2}\end{aligned}$$



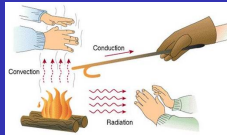
The exact solution for a convective boundary condition:

Temperature distribution

$$1 - \theta = \frac{T(x, t) - T_s}{T_\infty - T_s} = \operatorname{erfc} \left(\frac{x}{2\sqrt{\alpha t}} \right) - \exp \left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2} \right) \operatorname{erfc} \left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k} \right)$$

The complementary error function, $\operatorname{erfc} \xi$ is defined as $\operatorname{erfc} \xi = 1 - \operatorname{erf} \xi$. The **Gaussian error function**, $\operatorname{erf} \xi$, is a standard mathematical function that is tabulated.

Heat and Mass Transfer



Multidimensional Steady-State Conduction

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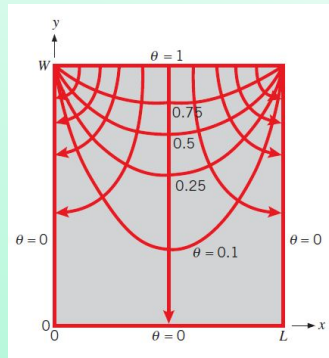
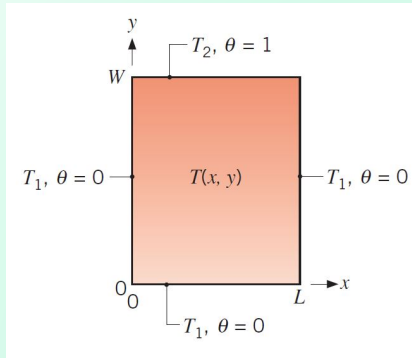
Governing equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

- Solve for $T(x, y)$
- Determine q''_x and q''_y from the rate equations

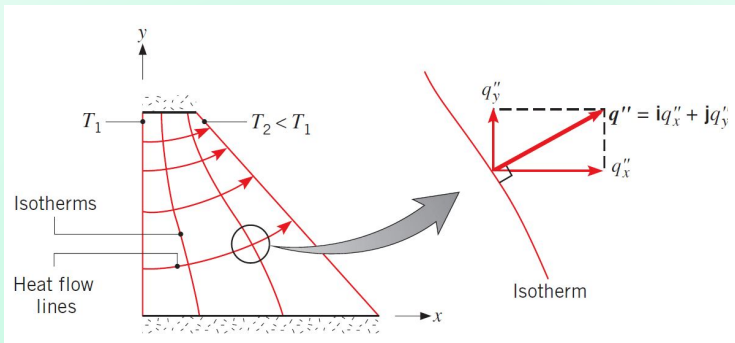
Methodologies/Approaches

- Analytical
- Graphical
- Numerical



Method of separation of variables:

$$\theta(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n} \sin \frac{n\pi x}{L} \frac{\sinh(n\pi y/L)}{\sinh(n\pi W/l)}$$



Conduction Shape Factor

$$q = kS\Delta T_{1-2}$$

where ΔT_{1-2} is the temp. difference between boundaries.

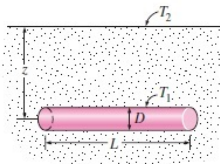
S (m) depends on the *geometry* of the system only and $R = 1/Sk$.

Conduction Shape Factor



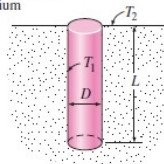
- (1) Isothermal cylinder of length L
buried in a semi-infinite medium
($L \gg D$ and $z > 1.5D$)

$$S = \frac{2\pi L}{\ln(4z/D)}$$



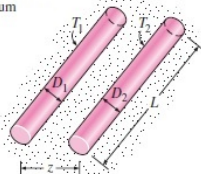
- (2) Vertical isothermal cylinder of length L
buried in a semi-infinite medium
($L \gg D$)

$$S = \frac{2\pi L}{\ln(4L/D)}$$



- (3) Two parallel isothermal cylinders
placed in an infinite medium
($L \gg D_1, D_2, z$)

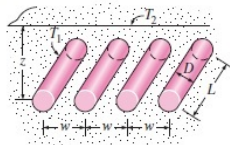
$$S = \frac{2\pi L}{\cosh^{-1}\left(\frac{4z^2 - D_1^2 - D_2^2}{2D_1D_2}\right)}$$



- (4) A row of equally spaced parallel isothermal
cylinders buried in a semi-infinite medium
($L \gg D, z$ and $w > 1.5D$)

$$S = \frac{2\pi L}{\ln\left(\frac{2w}{\pi D} \sinh \frac{2\pi z}{w}\right)}$$

(per cylinder)

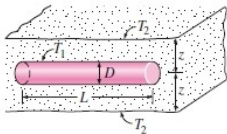


Conduction Shape Factor



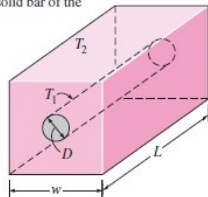
- (5) Circular isothermal cylinder of length L in the midplane of an infinite wall ($z > 0.5D$)

$$S = \frac{2\pi L}{\ln(8z/\pi D)}$$



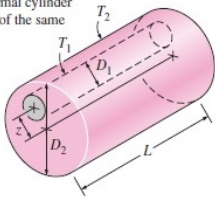
- (6) Circular isothermal cylinder of length L at the center of a square solid bar of the same length

$$S = \frac{2\pi L}{\ln(1.08w/D)}$$



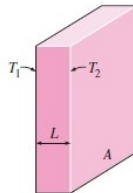
- (7) Eccentric circular isothermal cylinder of length L in a cylinder of the same length ($L > D_2$)

$$S = \frac{2\pi L}{\cosh^{-1}\left(\frac{D_1^2 + D_2^2 - 4z^2}{2D_1 D_2}\right)}$$



- (8) Large plane wall

$$S = \frac{A}{L}$$

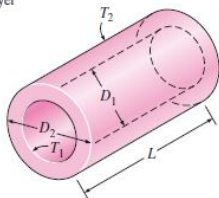


Conduction Shape Factor



(9) A long cylindrical layer

$$S = \frac{2\pi L}{\ln(D_2/D_1)}$$



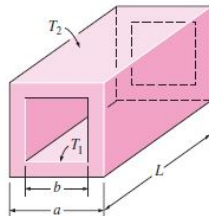
(10) A square flow passage

(a) For $a/b > 1.4$,

$$S = \frac{2\pi L}{0.93 \ln(0.948 a/b)}$$

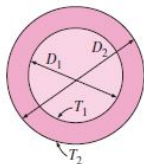
(b) For $a/b < 1.41$,

$$S = \frac{2\pi L}{0.785 \ln(a/b)}$$



(11) A spherical layer

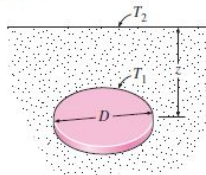
$$S = \frac{2\pi D_1 D_2}{D_2 - D_1}$$



(12) Disk buried parallel to the surface in a semi-infinite medium ($z \gg D$)

$$S = 4D$$

$$(S = 2D \text{ when } z = 0)$$

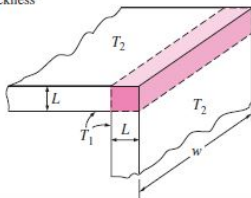


Conduction Shape Factor



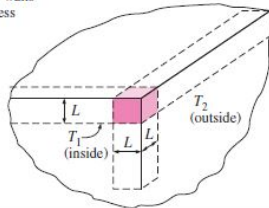
- (13) The edge of two adjoining walls of equal thickness

$$S = 0.54w$$



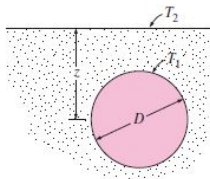
- (14) Corner of three walls of equal thickness

$$S = 0.15L$$



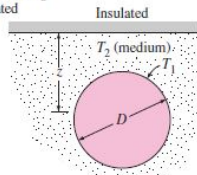
- (15) Isothermal sphere buried in a semi-infinite medium

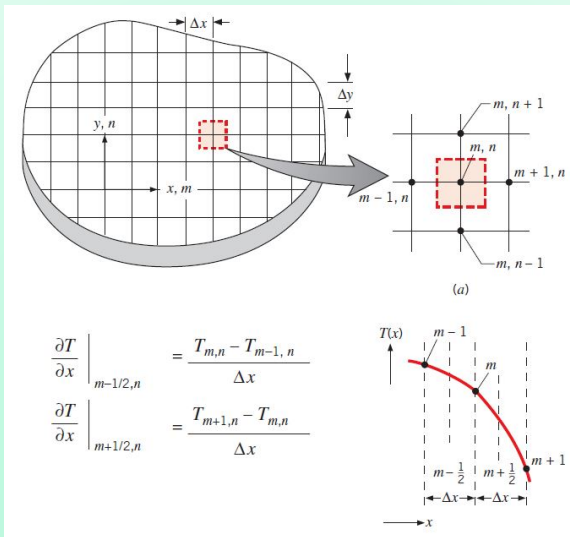
$$S = \frac{2\pi D}{1 - 0.25D/z}$$



- (16) Isothermal sphere buried in a semi-infinite medium at T_2 whose surface is insulated

$$S = \frac{2\pi D}{1 + 0.25D/z}$$

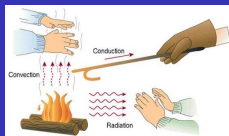




(a) Nodal Network

(b) Finite-difference approximation

Heat and Mass Transfer



Numerical Methods - FDS

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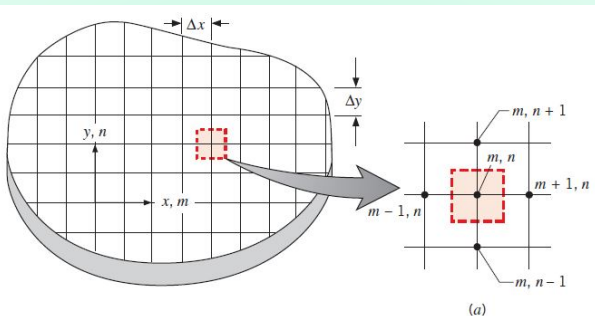


Advances in numerical computing now allow for complex heat transfer problems to be solved rapidly on computers. Some examples are:

- Finite-difference method
- Finite-element method
- Boundary-element method

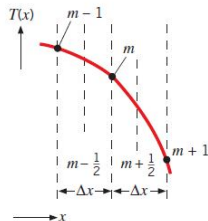
In general, these techniques are routinely used to solve problems in heat transfer, fluid dynamics, stress analysis, electrostatics and magnetics, etc. Finite-difference method is ease of application.

- Numerical techniques result in an **approximate solution**.
- Properties (e.g., T , u) are determined at **discrete** points in the region of interest - referred as **nodal points** or **nodes**.



$$\left. \frac{\partial T}{\partial x} \right|_{m-1/2, n} = \frac{T_{m, n} - T_{m-1, n}}{\Delta x}$$

$$\left. \frac{\partial T}{\partial x} \right|_{m+1/2, n} = \frac{T_{m+1, n} - T_{m, n}}{\Delta x}$$



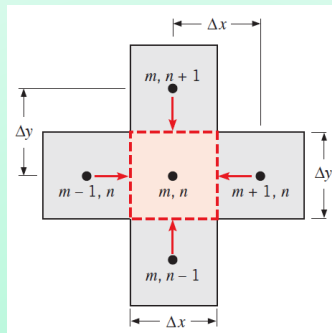
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad (1)$$

$$\begin{aligned} \left. \frac{\partial^2 T}{\partial x^2} \right|_{m,n} &\approx \frac{\left. \frac{\partial T}{\partial x} \right|_{m+1/2,n} - \left. \frac{\partial T}{\partial x} \right|_{m-1/2,n}}{\Delta x} \\ &\approx \frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{(\Delta x)^2} \end{aligned} \quad (2)$$

$$\left. \frac{\partial^2 T}{\partial y^2} \right|_{m,n} \approx \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{(\Delta y)^2} \quad (3)$$

Using a network for which $\Delta x = \Delta y$ and substituting Eqs. (2) and (3) in Eq. (1):

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$$



Assuming that all the heat flow is into the node, $\dot{E}_{in} + \dot{E}_g = 0$:

$$\sum_{i=1}^4 q_{(i) \rightarrow (m,n)} + \dot{q}(\Delta x \cdot \Delta y \cdot 1) = 0$$



$$q_{(m-1,n) \rightarrow (m,n)} = k(\Delta y \cdot 1) \frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

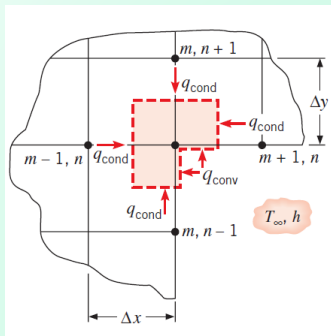
$$q_{(m+1,n) \rightarrow (m,n)} = k(\Delta y \cdot 1) \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$q_{(m,n+1) \rightarrow (m,n)} = k(\Delta x \cdot 1) \frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

$$q_{(m,n-1) \rightarrow (m,n)} = k(\Delta x \cdot 1) \frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} + \frac{\dot{q}(\Delta x)^2}{k} - 4T_{m,n} = 0$$

Energy Balance Method



$$q_{(m-1,n) \rightarrow (m,n)} = k(\Delta y \cdot 1) \frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$q_{(m,n+1) \rightarrow (m,n)} = k(\Delta x \cdot 1) \frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

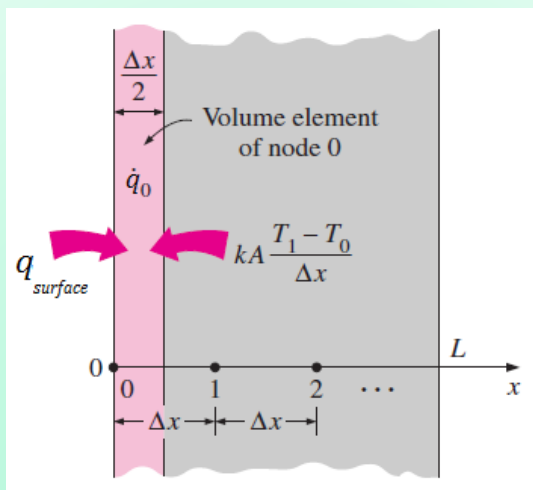
$$q_{(m+1,n) \rightarrow (m,n)} = k\left(\frac{\Delta y}{2} \cdot 1\right) \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$q_{(m,n-1) \rightarrow (m,n)} = k\left(\frac{\Delta x}{2} \cdot 1\right) \frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

$$q_{(\infty) \rightarrow (m,n)} = h \left(\frac{\Delta x}{2} \cdot 1 \right) (T_{\infty} - T_{m,n}) + h \left(\frac{\Delta y}{2} \cdot 1 \right) (T_{\infty} - T_{m,n})$$

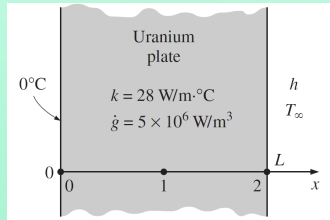
$$T_{m,n+1} + T_{m,n-1} + \frac{1}{2}(T_{m+1,n} + T_{m-1,n}) + \frac{h\Delta x}{k} T_{\infty} - \left(3 + \frac{h\Delta x}{k} \right) T_{m,n} = 0$$

Specified Heat Flux Boundary Condition



$$q_{\text{surface}} + kA \frac{T_1 - T_0}{\delta x} + \dot{q}A \frac{\Delta x}{2} = 0$$

Consider a large uranium plate of thickness $L = 4$ cm and $k = 28$ W/m K in which heat is generated uniformly at a constant rate of $\dot{q} = 5 \times 10^6$ W/m³. One side of the plate is maintained at 0°C by iced water while the other side is subjected to convection to an environment at $T_\infty = 30^\circ\text{C}$ with $h = 45$ W/m² K. Considering a total of three equally spaced nodes in the medium, two at the boundaries and one at the middle, estimate the exposed surface temperature of the plate under steady conditions using the finite difference approach.



For the interior node (1), the energy balance would result in:

$$\frac{T_0 - 2T_1 + T_2}{(\Delta x)^2} + \frac{\dot{q}}{k} = 0$$

$$T_0 - 2T_1 + T_2 = -\frac{\dot{q}(\Delta x)^2}{k}$$

$$2T_1 - T_2 = 71.43 \quad (1)$$

Let us write the governing equation for the corner Node (2):

$$h(T_\infty - T_2) + kA \frac{T_1 - T_2}{\Delta x} + \dot{q}A \frac{\Delta x}{2} = 0$$

$$T_1 - \left(1 + \frac{h\Delta x}{k}\right)T_2 = -\frac{h\Delta x}{k}T_\infty - \frac{\dot{q}(\Delta x)^2}{2k}$$

$$T_1 - 1.032T_2 = -36.68 \quad (2)$$



$$T_0 - 2T_1 + T_2 = -\frac{\dot{q}(\Delta x)^2}{k}$$

$$T_{m-1} - 2T_m + T_{m+1} = -\frac{\dot{q}(\Delta x)^2}{k}$$

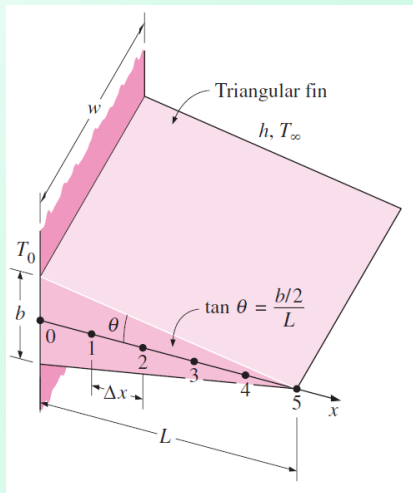
$$T_1 - \left(1 + \frac{h\Delta x}{k}\right)T_2 = -\frac{h\Delta x}{k}T_\infty - \frac{\dot{q}(\Delta x)^2}{2k}$$

$$T_{M-1} - \left(1 + \frac{h\Delta x}{k}\right)T_M = -\frac{h\Delta x}{k}T_\infty - \frac{\dot{q}(\Delta x)^2}{2k}$$



Consider an aluminum alloy fin ($k = 180 \text{ W/m K}$) of triangular cross section with length $L = 5 \text{ cm}$, base thickness $b = 1 \text{ cm}$, and very large width w in the direction normal to the plane of paper. The base of the fin is maintained at a temperature of $T_0 = 200^\circ\text{C}$. The fin is losing heat to the surrounding medium at $T_\infty = 25^\circ\text{C}$ with a heat transfer coefficient of $h = 15 \text{ W/m}^2 \text{ K}$. Using the finite difference method with six equally spaced nodes along the fin in the x -direction, determine (a) the temperatures at the nodes, (b) the rate of heat transfer from the fin for $w = 1 \text{ m}$, and (c) the fin efficiency.

Problem



$$T_0 = 200^\circ\text{C}$$

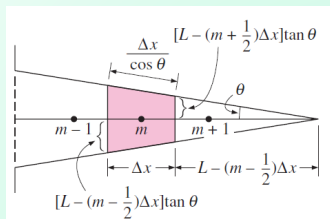
$$b = 1 \text{ cm}$$

$$T_\infty = 25^\circ\text{C}$$

$$k = 180 \text{ W/m K}$$

$$w = 1 \text{ cm}$$

$$h = 15 \text{ W/m}^2 \text{ K}$$



$$q_{\text{left}} + q_{\text{right}} + q_{\text{conv}} = 0$$

$$kA_{\text{left}} \frac{T_{m-1} - T_m}{\Delta x} + kA_{\text{right}} \frac{T_{m+1} - T_m}{\Delta x} + hA_{\text{conv}}(T_{\infty} - T_m) = 0$$

$$A_{\text{left}} = 2w[L - (m - 1/2)\Delta x] \tan \theta$$

$$A_{\text{right}} = 2w[L - (m + 1/2)\Delta x] \tan \theta$$

$$A_{\text{conv}} = 2w(\Delta x / \cos \theta)$$

$$\tan \theta = \frac{b/2}{L} = 0.1 \implies \theta = 5.71^\circ$$

$$(5.5 - m)T_{m-1} - (10.01 - 2m)T_m + (4.5 - m)T_{m+1} = -0.209$$

Four equations for $m = 1 \rightarrow 4$. Boundary condition at node (5):

$$kA_{\text{left}} \frac{T_4 - T_5}{\Delta x} + hA_{\text{conv}}(T_\infty - T_5)$$

$$A_{\text{left}} = 2w(\Delta x/2) \tan \theta \quad A_{\text{conv}} = 2w \frac{\Delta x/2}{\cos \theta}$$

Total 5 equations with 5 unknowns.

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} -8.01 & 3.5 & 0 & 0 & 0 \\ 3.5 & -6.01 & 2.5 & 0 & 0 \\ 0 & 2.5 & -4.01 & 1.5 & 0 \\ 0 & 0 & 1.5 & -2.01 & 0.5 \\ 0 & 0 & 0 & 1 & -1.01 \end{bmatrix}^{-1} \times \begin{bmatrix} -900.21 \\ -0.21 \\ -0.21 \\ -2.1 \\ -0.21 \end{bmatrix}$$
$$= \begin{bmatrix} 198.6 \\ 197.1 \\ 195.7 \\ 194.3 \\ 192.9 \end{bmatrix} \text{ } ^\circ\text{C}$$



Direct Methods

Solve in a systematic manner following a series of well-defined steps.

Iterative Methods

Start with an initial guess for the solution, and iterate until solution converges.



Application of the Gauss-Seidel iterative method to the finite difference equations in the previous triangular fin example.

Finite difference equations in explicit form

$$T_1 = 0.4371T_2 + 112.4137$$

$$T_2 = 0.5826T_1 + 0.4161T_3 + 0.0348$$

$$T_3 = 0.6238T_2 + 0.3743T_4 + 0.0521$$

$$T_4 = 0.7470T_3 + 0.2490T_5 + 0.1041$$

$$T_5 = 0.9921T_4 + 0.2073$$

Iteration	Nodal Temperature, °C				
	T_1	T_2	T_3	T_4	T_5
Initial Guess	195.0	195.0	195.0	195.0	195.0
1	197.6	196.3	195.5	194.7	193.4
2	198.2	196.9	195.8	194.5	193.2
3	198.5	197.2	195.9	194.5	193.2
4	198.6	197.3	195.9	194.5	193.2
5	198.7	197.3	195.9	194.5	193.2
6	198.7	197.3	195.9	194.5	193.2
7	198.7	197.3	195.9	194.5	193.2



$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

The problem must be discretized in time.

$$t = p\Delta t$$

$$\left. \frac{\partial T}{\partial t} \right|_{m,n} \approx \frac{T_{m,n}^{p+1} - T_{m,n}^p}{\Delta t}$$

In explicit method of solution, the temperatures are evaluated at the previous (p) time - forward-difference approximation to the time derivative.

$$\begin{aligned} \frac{1}{\alpha} \frac{T_{m,n}^{p+1} - T_{m,n}^p}{\Delta t} &= \frac{T_{m+1,n}^p + T_{m-1,n}^p - 2T_{m,n}^p}{(\Delta x)^2} \\ &+ \frac{T_{m,n+1}^p + T_{m,n-1}^p - 2T_{m,n}^p}{(\Delta y)^2} \end{aligned}$$



Solving for the nodal temperature at the new $(p + 1)$ time and assuming that $\Delta x = \Delta y$, it follows that:

$$T_{m,n}^{p+1} = \text{Fo}(T_{m+1,n}^p + T_{m-1,n}^p + T_{m,n+1}^p + T_{m,n-1}^p) + (1 - 4\text{Fo})T_{m,n}^p$$

$$\text{where } \text{Fo} = \frac{\alpha \Delta t}{(\Delta x)^2}$$

If the system is 1D in x , the explicit form of the finite-difference equation for an interior node m reduces to:

$$T_m^{p+1} = \text{Fo}(T_{m+1}^p + T_{m-1}^p) + (1 - 2\text{Fo})T_m^p$$



- Explicit method is not unconditionally stable.
- The solution may be characterized by numerically induced oscillations, which are physically impossible
- Oscillations may become *unstable*, causing the solution to diverge from the actual steady-state conditions.

Stability Criterion

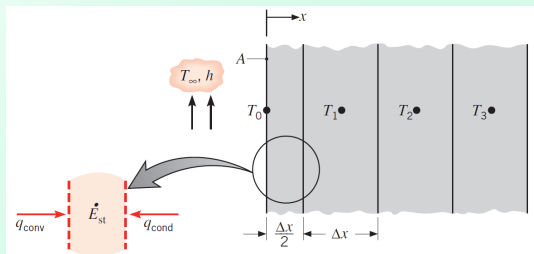
The criterion is determined by requiring that the coefficient associated with the node of interest at the previous time is greater than or equal to zero.

$$T_m^{p+1} = \text{Fo}(T_{m+1}^p + T_{m-1}^p) + (1 - 2\text{Fo})T_m^p$$

$$1\text{D} : (1 - 2\text{Fo}) \geq 0 \implies \text{Fo} \leq \frac{1}{2}$$

$$2\text{D} : (1 - 4\text{Fo}) \geq 0 \implies \text{Fo} \leq \frac{1}{4}$$

Energy Balance at the Boundary



$$\dot{E}_{in} + \dot{E}_g = \dot{E}_{st}$$

$$hA(T_{\infty} - T_0^p) + \frac{kA}{\Delta x}(T_1^p - T_0^p) = \rho A \frac{\Delta x}{2} C_p \frac{T_0^{p+1} - T_0^p}{\Delta t}$$

$$T_0^{p+1} = 2 \text{Fo} (T_1^p + \text{Bi} T_{\infty}) + (1 - 2 \text{Fo} - 2 \text{Bi} \text{Fo}) T_0^p$$

$$\text{Bi} = \frac{h\Delta x}{k}$$

From the stability Criteria:

$$1 - 2 \text{Fo} - 2 \text{Bi} \text{Fo} \geq 0$$

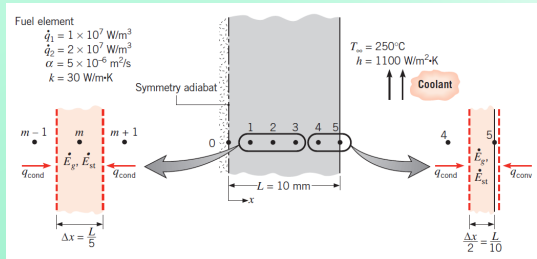
$$\boxed{\text{Fo}(1 + \text{Bi}) \leq \frac{1}{2}}$$

Problem



A fuel element of a nuclear reactor is in the shape of a plane wall of thickness $2L = 20$ mm and is convectively cooled at both surfaces, with $h = 1100$ W/m² K and $T_\infty = 250^\circ\text{C}$. At normal operating power, heat is generated uniformly within the element at a volumetric rate of $\dot{q}_1 = 10^7$ W/m³. A departure from the steady-state conditions associated with normal operation will occur if there is a change in the generation rate. Consider a sudden change to $\dot{q}_2 = 2 \times 10^7$ W/m³, and determine the fuel element temperature distribution after 1.5 s.

The fuel element thermal properties are $k = 30$ W/m K and $\alpha = 5 \times 10^{-6}$ m²/s.



For $m = 0 \rightarrow 4$:

$$kA \frac{T_{m-1}^p - T_m^p}{\Delta x} + kA \frac{T_{m+1}^p - T_m^p}{\Delta x} + \dot{q}A\Delta x = \rho A\Delta x C_p \frac{T_m^{p+1} - T_m^p}{\Delta t}$$

$$T_m^{p+1} = \text{Fo} \left[T_{m-1}^p + T_{m+1}^p + \frac{\dot{q}(\Delta x)^2}{k} \right] + (1 - 2\text{Fo})T_m^p \quad m=0 \rightarrow 4$$

For $m = 5$:

$$kA \frac{T_4^p - T_5^p}{\Delta x} + hA(T_\infty - T_5^p) + \dot{q}A \frac{\Delta x}{2} = \rho A \frac{\Delta x}{2} C_p \frac{T_5^{p+1} - T_5^p}{\Delta t}$$

$$T_5^{p+1} = 2\text{Fo} \left[T_4^p + \text{Bi} T_\infty + \frac{\dot{q}(\Delta x)^2}{2k} \right] + (1 - 2\text{Fo} - 2\text{Bi} \text{Fo})T_5^p \quad m=5$$

$$\text{Fo}(1 + \text{Bi}) \leq \frac{1}{2}$$

$$\text{Bi} = 0.0733 \implies \text{Fo} \leq 0.466 \implies \Delta t \leq 0.373 \text{ s}$$

Assuming $\Delta t = 0.3$ s, $Fo = 0.375$,

$$T_0^{p+1} = 0.375(2T_1^p + 2.67) + 0.250T_0^p$$

$$T_1^{p+1} = 0.375(T_0^p + T_2^p + 2.67) + 0.250T_0^p$$

$$T_2^{p+1} = 0.375(T_1^p + T_3^p + 2.67) + 0.250T_0^p$$

$$T_3^{p+1} = 0.375(T_2^p + T_4^p + 2.67) + 0.250T_0^p$$

$$T_4^{p+1} = 0.375(T_3^p + T_5^p + 2.67) + 0.250T_0^p$$

$$T_5^{p+1} = 0.750(T_4^p + 19.67) + 0.195T_0^p$$

For 1D, steady state, \dot{q} , symmetrical about the plane:

$$T(x) = \frac{\dot{q}L^2}{2k} \left(1 - \frac{x^2}{L^2}\right) + T_s \qquad T_5^{p=0} = 250 + \frac{10^7 \times 0.01}{1100} = 340.9^\circ\text{C}$$

$$T_s = T_\infty + \frac{\dot{q}L}{h} \qquad T_m^{p=0} = 16.67(1 - 10000x^2) + 340.9$$



p	$t(s)$	T_0	T_1	T_2	T_3	T_4	T_5
0	0	357.58	356.91	354.91	351.58	346.91	340.91
1	0.3	358.08	357.41	355.41	352.08	347.41	341.41
2	0.6	358.58	357.91	355.91	352.58	347.91	341.88
3	0.9	359.08	358.41	356.41	353.08	348.41	342.35
4	1.2	359.58	358.91	356.91	353.58	348.89	342.82
5	1.5	360.08	359.41	357.41	354.07	349.37	343.27
∞	∞	465.15	463.82	459.82	453.15	443.82	431.82



$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

$$\left. \frac{\partial T}{\partial t} \right|_{m,n} \approx \frac{T_{m,n}^{p+1} - T_{m,n}^p}{\Delta t}$$

In **implicit method** of solution, the temperatures are evaluated at the **new** ($p + 1$) time - **backward-difference** approximation to the time derivative.

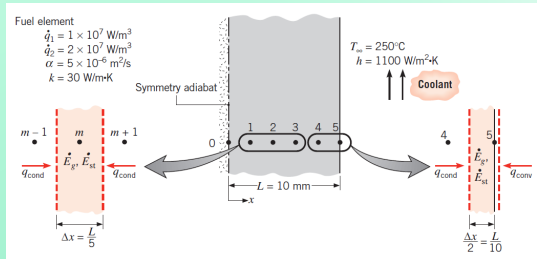
$$\begin{aligned} \frac{1}{\alpha} \frac{T_{m,n}^{p+1} - T_{m,n}^p}{\Delta t} &= \frac{T_{m+1,n}^{p+1} + T_{m-1,n}^{p+1} - 2T_{m,n}^{p+1}}{(\Delta x)^2} \\ &+ \frac{T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1} - 2T_{m,n}^{p+1}}{(\Delta y)^2} \end{aligned}$$

Problem



A fuel element of a nuclear reactor is in the shape of a plane wall of thickness $2L = 20$ mm and is convectively cooled at both surfaces, with $h = 1100$ W/m² K and $T_\infty = 250^\circ\text{C}$. At normal operating power, heat is generated uniformly within the element at a volumetric rate of $\dot{q}_1 = 10^7$ W/m³. A departure from the steady-state conditions associated with normal operation will occur if there is a change in the generation rate. Consider a sudden change to $\dot{q}_2 = 2 \times 10^7$ W/m³, and determine the fuel element temperature distribution after 1.5 s.

The fuel element thermal properties are $k = 30$ W/m K and $\alpha = 5 \times 10^{-6}$ m²/s.



For $m = 0 \rightarrow 4$:

$$kA \frac{T_{m-1}^{p+1} - T_m^{p+1}}{\Delta x} + kA \frac{T_{m+1}^{p+1} - T_m^{p+1}}{\Delta x} + \dot{q}A\Delta x = \rho A\Delta x C_p \frac{T_m^{p+1} - T_m^p}{\Delta t}$$

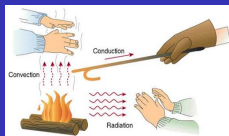
$$\boxed{\text{Fo} T_m^{p+1} - (1 + 2\text{Fo})T_m^{p+1} + \text{Fo}T_{m+1}^{p+1} = -T_m^p - \text{Fo} \frac{\dot{q}(\Delta x)^2}{k}}_{m=0 \rightarrow 4}$$

For $m = 5$:

$$kA \frac{T_4^{p+1} - T_5^{p+1}}{\Delta x} + hA(T_\infty - T_5^{p+1}) + \dot{q}A \frac{\Delta x}{2} = \rho A \frac{\Delta x}{2} C_p \frac{T_5^{p+1} - T_5^p}{\Delta t}$$

$$\boxed{2\text{Fo}T_4^{p+1} - (1 + 2\text{Fo} + 2\text{Bi Fo})T_5^{p+1} = -T_5^p - \text{Fo} \frac{\dot{q}(\Delta x)^2}{k}}_{m=5}$$

Heat and Mass Transfer



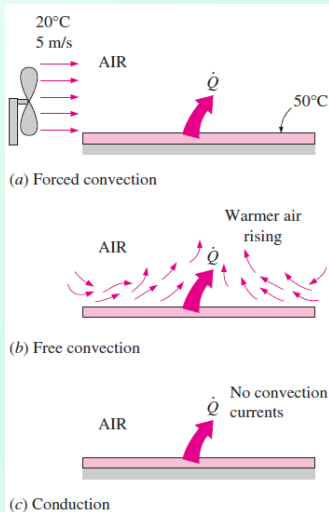
Introduction to Convection

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Convective heat transfer involves

- fluid motion
- heat conduction

The fluid motion enhances the heat transfer, since it brings hotter and cooler chunks of fluid into contact, initiating higher rates of conduction at a greater number of sites in fluid.

Therefore, the rate of heat transfer through a fluid is much higher by convection than it is by conduction.

Higher the fluid velocity, the higher the rate of heat transfer.



Convection heat transfer strongly depends on

- fluid properties: μ, k, ρ, C_p
- fluid velocity: V
- geometry and the roughness of the solid surface
- type of fluid flow (laminar or turbulent)

Newton's law of cooling

$$q_{conv} = hA_s (T_s - T_\infty)$$

T_∞ is the temp. of the fluid sufficiently far from the surface



Local heat flux

$$q''_{conv} = h_l (T_s - T_\infty)$$

h_l is the local convection coefficient

Flow conditions vary on the surface: q'' , h vary along the surface.

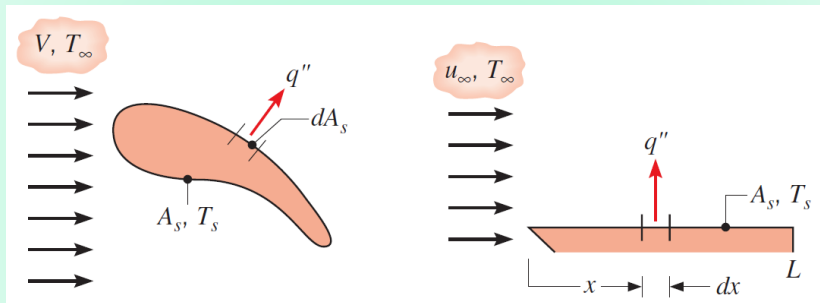
The total heat transfer rate q :

$$\begin{aligned} q_{conv} &= \int_{A_s} q'' dA_s \\ &= (T_s - T_\infty) \int_{A_s} h dA_s \end{aligned}$$

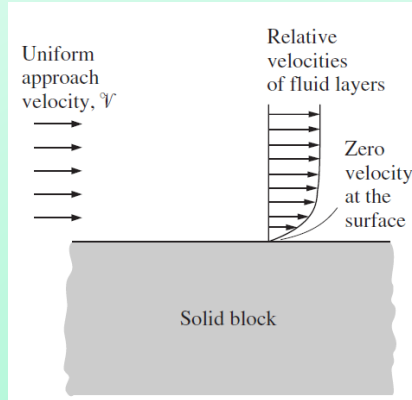
Defining an *average convection coefficient* \bar{h} for the entire surface,

$$q_{conv} = \bar{h}A_s (T_s - T_\infty)$$

$$\bar{h} = \frac{1}{A_s} \int_{A_s} h dA_s$$



A fluid flowing over a stationary surface - **no-slip condition**



A fluid and a solid surface will have the same T at the point of contact, known as **no-temperature-jump condition**.



With no-slip and the no-temperature-jump conditions: the heat transfer from the solid surface to the fluid layer adjacent to the surface is by **pure conduction**.

$$q''_{conv} = q''_{cond} = -k_{fluid} \left. \frac{\partial T}{\partial y} \right|_{y=0}$$

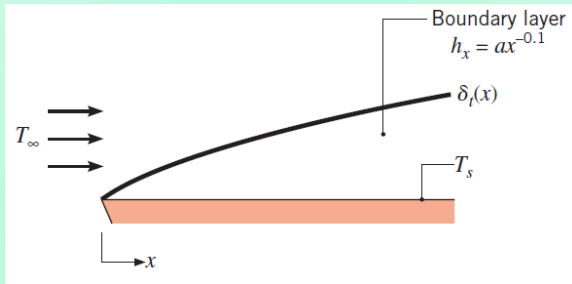
T represents the temperature distribution in the fluid $(\partial T / \partial y)_{y=0}$
i.e., the temp. gradient at the surface.

$$q''_{conv} = h(T_s - T_\infty)$$

$$h = \frac{-k_{fluid} \left(\frac{\partial T}{\partial y} \right)_{y=0}}{T_s - T_\infty}$$

Experimental results for the local heat transfer coefficient h_x for flow over a flat plate with an extremely rough surface were found to fit the relation $h_x(x) = x^{-0.1}$ where x (m) is the distance from the leading edge of the plate.

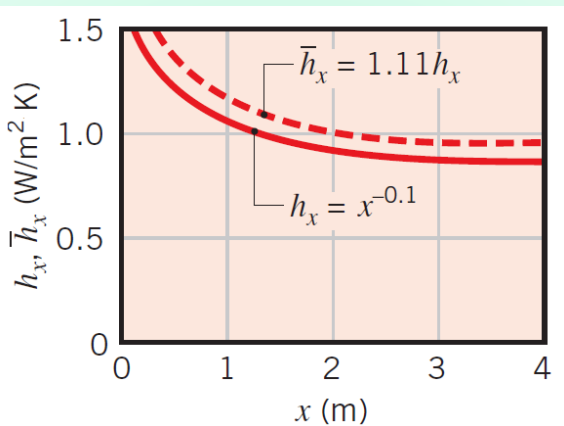
- Develop an expression for the ratio of the average heat transfer coefficient \bar{h}_x for a path of length x to the local heat transfer coefficient h_x at x .
- Plot the variation of h_x and \bar{h}_x as a function of x .



The average value of h over the region from 0 to x is:

$$\begin{aligned}\bar{h}_x &= \frac{1}{x} \int_0^x h_x(x) dx \\ &= \frac{1}{x} \int_0^x x^{-0.1} dx \\ &= \frac{1}{x} \frac{x^{0.9}}{0.9} = 1.11x^{-0.1}\end{aligned}$$

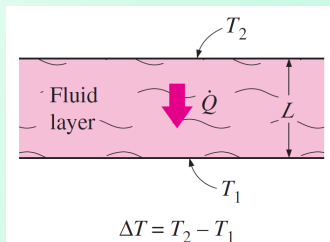
$$\boxed{\bar{h}_x = 1.11h_x}$$



Comments

Boundary layer development causes both h_l and \bar{h} to decrease with increasing distance from the leading edge. The average coefficient up to x must therefore exceed the local value at x .

$$\text{Nu} = \frac{hL_c}{k_{\text{fluid}}}$$



Heat transfer through the fluid layer will be by **convection** when the fluid involves some motion and by **conduction** when the fluid layer is motionless.

$$q_{\text{conv}} = h\Delta T \quad q_{\text{cond}} = k \frac{\Delta T}{L}$$

$$\frac{q_{\text{conv}}}{q_{\text{cond}}} = \frac{h\Delta T}{k\Delta T/L} = \frac{hL}{k} = \text{Nu}$$



$$\text{Nu} = \frac{q_{conv}}{q_{cond}}$$

Nusselt number: enhancement of heat transfer through a fluid layer as a result of convection relative to conduction across the same fluid layer.

$\text{Nu} \gg 1$ for a fluid layer - the more effective the convection

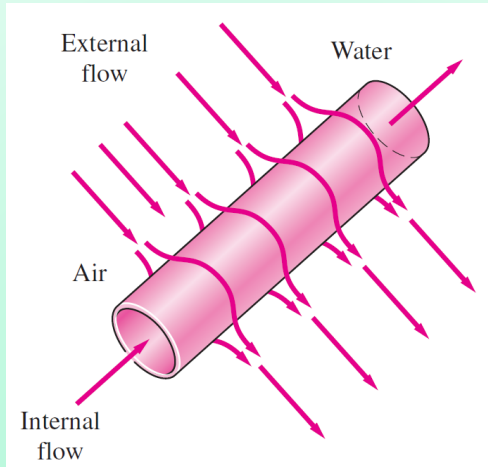
$\text{Nu} = 1$ for a fluid layer - heat transfer across the layer is by **pure conduction**

$\text{Nu} < 1$???



- German engineer, born in Germany (1882)
- Doctoral thesis - Conductivity of Insulating Materials
- Prof. - Heat and Momentum Transfer in Tubes
- 1915 - pioneering work in basic laws of transfer
 - Dimensionless groups - similarity theory of heat transfer
 - Film condensation of steam on vertical surfaces
 - Combustion of pulverized coal
 - Analogy of heat transfer and mass transfer in evaporation

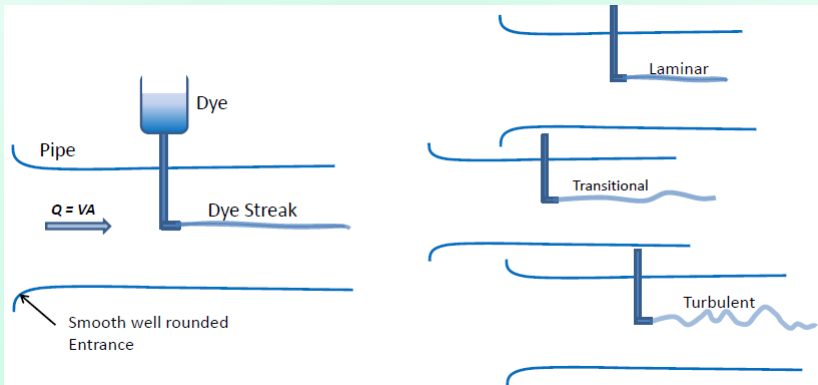
Worked till 70 years. Lived for 75 years and died in München on September 1, 1957.



External - flow of an unbounded fluid over a surface

Internal - flow is completely bounded by solid surfaces

Laminar and Turbulent Flows



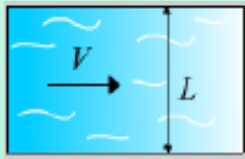
Laminar - smooth and orderly: flow of **high-viscosity fluids** such as oils at **low velocities**

Internal - chaotic and highly disordered fluid motion: flow of **low-viscosity fluids** such as air at **high velocities**

The flow regime greatly influences the heat transfer rates and the required power for pumping.

Osborne Reynolds in 1880's, discovered that the flow regime depends mainly on the ratio of the **inertia forces** to **viscous forces** in the fluid.

Re can be viewed as the ratio of the inertia forces to the viscous forces acting on a fluid volume element.



$$\text{Re} = \frac{\text{Inertia forces}}{\text{Viscous}} = \frac{VL_c}{\nu} = \frac{\rho VL_c}{\mu}$$



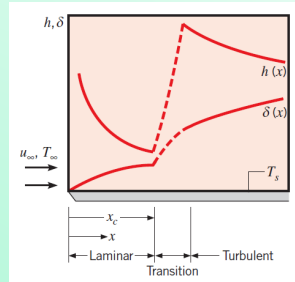
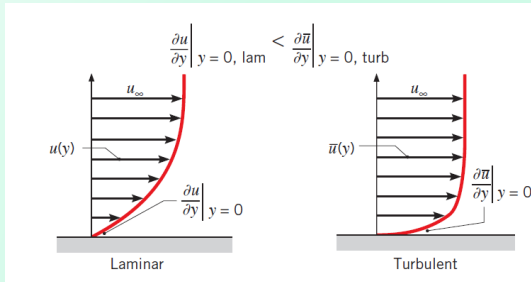
Taylor and Von Karman (1937)

Turbulence is an irregular motion which in general makes its appearance in fluids, gaseous or liquids, when they flow past solid surfaces or even when neighboring streams of same fluid past over one another.

Turbulent fluid motion is an irregular condition of flow in which various quantities show a random variation with time and space coordinates, so that statistically distinct average values can be discerned.

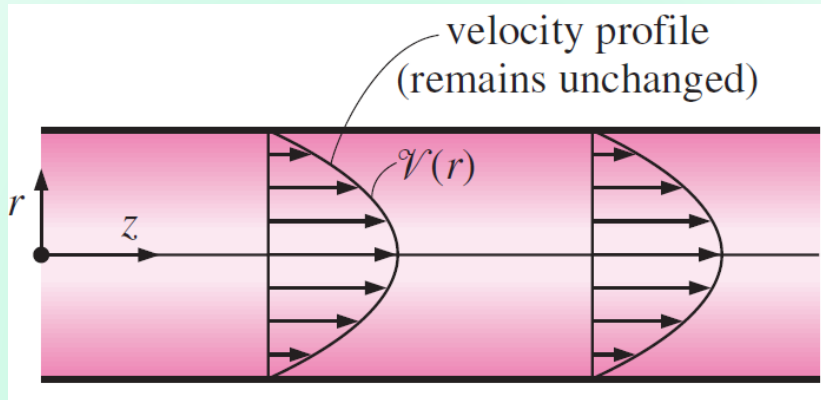
The Effects of Turbulence

Because of the motion of **eddies**, the transport of momentum, energy, and species is enhanced.



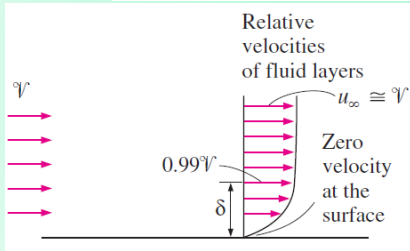
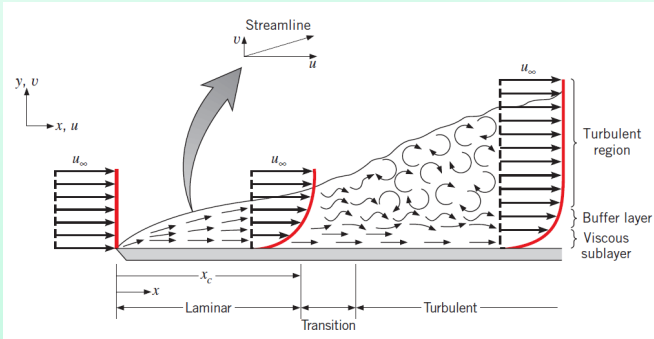
The velocity gradient at the surface, and therefore the surface shear stress, is much larger for δ_{turb} than for δ_{lam} . Similarly for temp. & conc. gradients.

Turbulence is desirable. However, the increase in wall shear stress will have the adverse effect of increasing pump or fan power.



1-D flow in a circular pipe

Velocity Boundary Layer



$$V_{x=\delta} = 0.99U_{\infty}$$

Friction force per unit area is called **shear stress**

Surface shear stress

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0}$$

The determination of τ_w is not practical as it requires a knowledge of the flow velocity profile. A more practical approach in external flow is to relate τ_w to the upstream velocity U_∞ as:

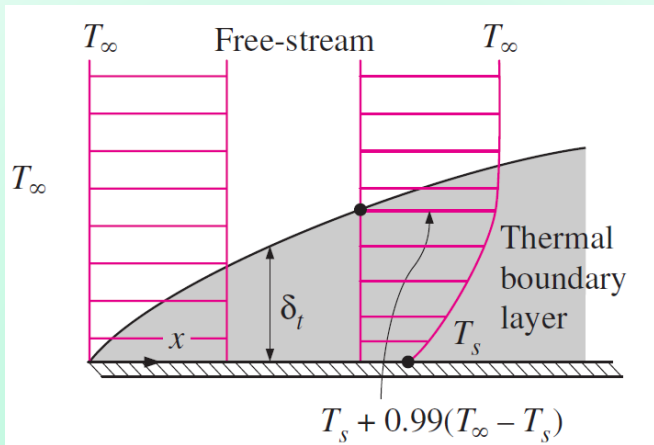
Skin friction coefficient

$$\tau_w = C_f \frac{\rho U_\infty^2}{2}$$

Friction force over the entire surface

$$F_f = C_f A_s \frac{\rho U_\infty^2}{2}$$

Thermal Boundary Layer



δ_t at any location along the surface at which
 $(T - T_s) = 0.99(T_\infty - T_s)$



- Shape of the temp. profile in the thermal boundary layer dictates the convection heat transfer between a solid surface and the fluid flowing over it.
- In flow over a heated (or cooled) surface, both velocity and thermal boundary layers will develop simultaneously.
- Noting that the fluid velocity will have a strong influence on the temp. profile, the development of the velocity boundary layer relative to the thermal boundary layer will have a strong effect on the convection heat transfer.

$$\text{Pr} = \frac{\text{Molecular diffusivity of momentum}}{\text{Molecular diffusivity of heat}} = \frac{\nu}{\alpha} = \frac{\mu C_p}{k}$$



Typical ranges of Pr for common fluids

Fluid	Pr
Liquid metals	0.004-0.030
Gases	0.7-1.0
Water	1.7-13.7
Light organic fluids	5-50
Oils	50-100,000
Glycerin	2000-100,000



$$\frac{\delta}{\delta_t} \approx \text{Pr}^n$$

n is positive exponent

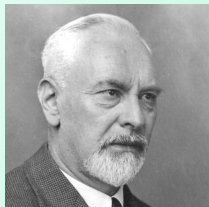
- $\text{Pr} \cong 1$ for gases \implies both momentum and heat dissipate through the fluid at about the same rate.
- Heat diffuses very quickly in liquid metals ($\text{Pr} < 1$).
- Heat diffuses very slowly in oils ($\text{Pr} > 1$) relative to momentum.
- Therefore, thermal boundary layer is much thicker for liquid metals and much thinner for oils relative to the velocity boundary layer.

$$\delta = \delta_t \text{ for } \text{Pr} = 1$$

$$\delta > \delta_t \text{ for } \text{Pr} > 1$$

$$\delta < \delta_t \text{ for } \text{Pr} < 1$$

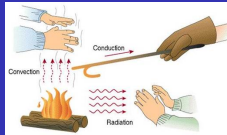
$$\text{Pr} = \frac{\nu}{\alpha}$$



- German Physicist, born in Bavaria (1875 - 1953)
- Father of aerodynamics
- Prof. of Applied Mechanics at Göttingen for 49 years (until his death)
- His work in fluid dynamics is still used today in many areas of aerodynamics and chemical engineering.

His discovery in 1904 of the **Boundary Layer** which adjoins the surface of a body moving in a fluid led to an understanding of skin friction drag and of the way in which streamlining reduces the drag of airplane wings and other moving bodies.

Heat and Mass Transfer



Convection Equations

Sudheer Siddapureddy

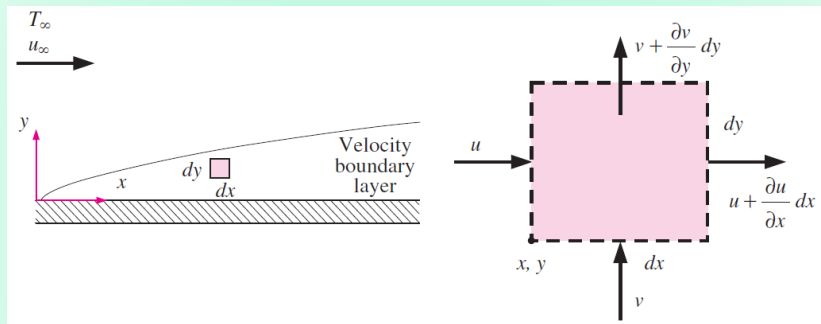
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Assuming the flow/fluid to be:

- 2D, Steady
- Newtonian
- constant properties (ρ, μ, k , etc.)





Rate of mass flow into CV = Rate of mass flow out of CV

rate of fluid entering CV_{left}:

$$\rho u(dy \cdot 1)$$

rate of fluid leaving CV_{right}:

$$\rho(u + \frac{\partial u}{\partial x} dx)(dy \cdot 1)$$

$$\rho u(dy \cdot 1) + \rho v(dx \cdot 1) = \rho(u + \frac{\partial u}{\partial x} dx)(dy \cdot 1) + \rho(v + \frac{\partial v}{\partial y} dy)(dx \cdot 1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$



Expressing Newton's second law of motion for the control volume:

(Mass) (Acceleration in x) = (Net body and surface forces in x)

$$\delta m \cdot a_x = F_{\text{surface},x} + F_{\text{body},x} \quad (3)$$

$$\delta m = \rho(dx \cdot dy \cdot 1)$$

$$\begin{aligned} a_x &= \frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} \\ &= u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \end{aligned} \quad (4)$$

Steady state doesn't mean that acceleration is zero.

Ex: Garden hose nozzle



Body forces: gravity, electric and magnetic forces - \propto volume.

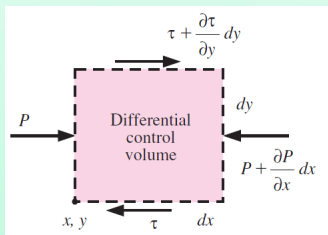
Surface forces: pressure forces due to hydrostatic pressure and shear stresses due to viscous effects - \propto surface area.

Viscous forces has two components:

- 1 normal to the surface- **normal stress**
related to velocity gradients $\partial u/\partial x$ and $\partial v/\partial y$
- 2 along the wall surface - **shear stress**
related to $\partial u/\partial y$

For simplicity, the normal stresses are neglected.

Momentum Equation



$$\begin{aligned} F_{\text{surface},x} &= \left(\frac{\partial \tau}{\partial y} dy \right) (dx \cdot 1) - \left(\frac{\partial P}{\partial x} dx \right) (dy \cdot 1) \\ &= \left(\frac{\partial \tau}{\partial y} - \frac{\partial P}{\partial x} \right) (dx \cdot dy \cdot 1) \end{aligned}$$

$$\tau = \mu \left(\frac{\partial u}{\partial y} \right)$$

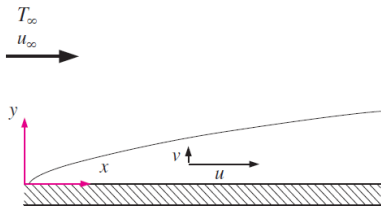
$$= \left(\mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x} \right) (dx \cdot dy \cdot 1) \quad (5)$$



Combining Eqs. (3), (4) and (5):

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x}$$

x-momentum equation



1) Velocity components:

$$v \ll u$$

2) Velocity gradients:

$$\frac{\partial v}{\partial x} \approx 0, \frac{\partial v}{\partial y} \approx 0$$

$$\frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial y}$$

3) Temperature gradients:

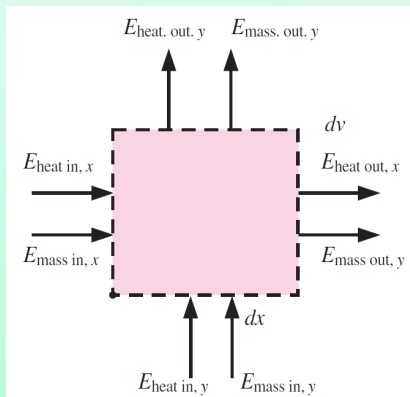
$$\frac{\partial T}{\partial x} \ll \frac{\partial T}{\partial y}$$

$$\frac{\partial P}{\partial y} = 0$$

y-momentum equation

$$P = P(x) \implies \frac{\partial P}{\partial x} = \frac{dP}{dx}$$

Energy Equation: Balance



$$(\dot{E}_{in} - \dot{E}_{out})_{\text{by mass}} + (\dot{E}_{in} - \dot{E}_{out})_{\text{by heat}} + (\dot{E}_{in} - \dot{E}_{out})_{\text{by work}} = 0 \quad (6)$$

Flowing fluid stream: is associated with enthalpy (internal energy and flow energy), potential energy (PE) and kinetic energy (KE)

The total energy of a flowing stream is:

$$\dot{m}(h + pe + ke) = \dot{m} \left(h + \cancel{gz} + \frac{\cancel{u^2 + v^2}}{2} \right) = \dot{m}C_pT$$

KE: $[m^2/s^2] \approx J/kg$. h is in kJ/kg . So, KE is expressed in kJ/kg by dividing it by 1000. KE term at low velocities is negligible.

By similar argument, **PE** term is negligible.

$$\begin{aligned} (\dot{E}_{in} - \dot{E}_{out})_{by\ mass,x} &= (\dot{m}C_pT)_x - \left[(\dot{m}C_pT)_x + \frac{\partial(\dot{m}C_pT)_x}{\partial x} dx \right] \\ &= - \frac{\partial[\rho u(dy \cdot 1)C_pT]}{\partial x} dx \end{aligned}$$

$$\boxed{(\dot{E}_{in} - \dot{E}_{out})_{by\ mass} = -\rho C_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) dx dy} \quad (7)$$



$$\begin{aligned}(\dot{E}_{in} - \dot{E}_{out})_{\text{by heat},x} &= q_x - \left(q_x + \frac{\partial q_x}{\partial x} dx \right) \\ &= -\frac{\partial}{\partial x} \left(-k(dy \cdot 1) \frac{\partial T}{\partial x} \right) dx \\ &= k \frac{\partial^2 T}{\partial x^2} dx dy\end{aligned}$$

$$\boxed{(\dot{E}_{in} - \dot{E}_{out})_{\text{by heat}} = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) dx dy} \quad (8)$$



- Work done by body forces: considered only if significant gravitational, electric, or magnetic effects exist.
- Work done by surface faces: consists of forces due to fluid pressure and viscous shear stresses.
 - Work done by pressure (the flow work) is accounted in enthalpy
 - Shear stresses that result from viscous effects are usually small (low velocities)



Combining Eqs. (6), (7), and (8):

$$\underbrace{\rho C_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right)}_{\text{convection}} = k \underbrace{\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)}_{\text{conduction}}$$

Net energy convected by the fluid out of CV

=

Net energy transferred into CV by conduction



When viscous shear stresses are not neglected, then:

$$\underbrace{\rho C_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right)}_{\text{convection}} = k \underbrace{\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)}_{\text{conduction}} + \underbrace{\mu \Phi}_{\text{viscous dissipation}}$$

where the **viscous dissipation term** is given as:

$$\mu \Phi = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + 2\mu \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right]$$

This accounts for the rate at which **mechanical work is irreversibly converted to thermal energy due to viscous effects in the fluid.**



$$\underbrace{u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}}_{\text{advection}} = \alpha \underbrace{\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)}_{\text{diffusion}}$$

When the fluid is stationary, $u = v = 0$:

$$k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) = 0$$



Steady incompressible, laminar flow of a fluid with constant properties:

$$\text{Continuity:} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\text{Momentum:} \quad \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu \frac{\partial^2 u}{\partial y^2} - \frac{dP}{dx}$$

$$\text{Energy:} \quad \rho C_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

with the boundary conditions

$$\text{At } x = 0 : \quad u(0, y) = u_\infty, \quad T(0, y) = T_\infty$$

$$\text{At } y = 0 : \quad u(x, 0) = 0, \quad v(x, 0) = 0, T(x, 0) = T_s$$

$$\text{At } y \rightarrow \infty : \quad u(x, \infty) = u_\infty, \quad T(x, \infty) = T_\infty$$



$$x^* = \frac{x}{L}, y^* = \frac{y}{L}, u^* = \frac{u}{U_\infty}, v^* = \frac{v}{U_\infty}, P^* = \frac{P}{\rho U_\infty^2}, T^* = \frac{T - T_s}{T_\infty - T_s}$$

Continuity:
$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

Momentum:
$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{\text{Re}_L} \frac{\partial^2 u^*}{\partial y^{*2}} - \frac{dP^*}{dx^*}$$

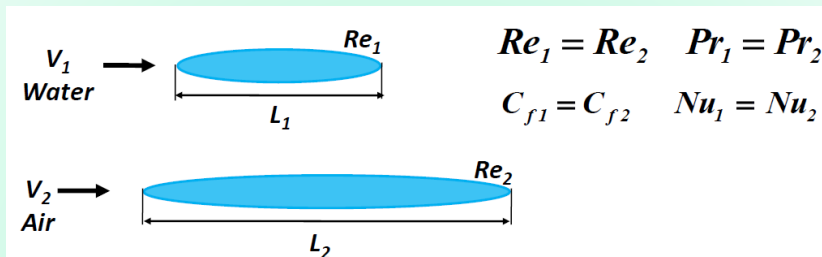
Energy:
$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{\text{Re}_L \text{Pr}} \frac{\partial^2 T^*}{\partial y^{*2}}$$

with the boundary conditions

At $x^* = 0$:
$$u^*(0, y^*) = 1, \quad T^*(0, y^*) = 1$$

At $y^* = 0$:
$$u^*(x^*, 0) = 0, \quad v^*(x^*, 0) = 0, T(x^*, 0) = 0$$

At $y^* \rightarrow \infty$:
$$u^*(x^*, \infty) = 1, \quad T^*(x^*, \infty) = 1$$



Parameters before nondimensionalizing: $L, V, T_\infty, \nu, \alpha$

Parameters after nondimensionalizing: Re, Pr The number of parameters is reduced greatly by nondimensionalizing the convection equations.



For a given geometry, the solution for u^* can be expressed as:

$$u^* = f_1(x^*, y^*, \text{Re}_L)$$

The shear stress at the surface:

$$\begin{aligned}\tau_w &= \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} \\ &= \frac{\mu U_\infty}{L} \left. \frac{\partial u^*}{\partial y^*} \right|_{y^*=0} \\ &= \frac{\mu U_\infty}{L} f_2(x^*, \text{Re}_L)\end{aligned}$$



Now the local friction factor:

$$\begin{aligned}C_{f,x} &= \frac{\tau_w}{\rho U_\infty^2 / 2} \\ &= \frac{\mu U_\infty / L}{\rho U_\infty^2 / 2} f_2(x^*, \text{Re}_L) \\ &= \frac{2}{\text{Re}_L} f_2(x^*, \text{Re}_L)\end{aligned}$$

$$C_{f,x} = f_3(x^*, \text{Re}_L)$$

Friction coefficient for a given geometry can be expressed in terms of the Re and the the dimensionless space variable x^* alone, instead of being expressed in terms of x , L , V , ρ and μ .

This is a very significant finding, and shows the value of nondimensionalized equations.



Similarly,

$$T^* = g_1(x^*, y^*, \text{Re}_L, \text{Pr})$$

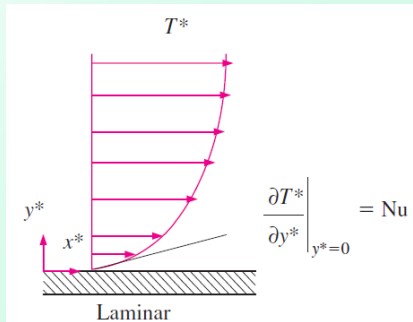
The local convective heat coefficient becomes:

$$h_x = -\frac{k}{T_s - T_\infty} \left. \frac{\partial T}{\partial y} \right|_{y=0} = -\frac{k(T_\infty - T_s)}{L(T_s - T_\infty)} \left. \frac{\partial T^*}{\partial y^*} \right|_{y^*=0} = \frac{k}{L} \left. \frac{\partial T^*}{\partial y^*} \right|_{y^*=0}$$

Nusselt number relation gives:

$$\text{Nu}_x = \frac{h_x L}{k} = \left. \frac{\partial T^*}{\partial y^*} \right|_{y^*=0} = g_2(x^*, \text{Re}_L, \text{Pr})$$

The Nu is equivalent to the dimensionless temp. gradient at the surface, and thus it is properly referred to as the dimensionless heat transfer coefficient.



The Nu is equivalent to the dimensionless temp. gradient at the surface, and thus it is properly referred to as the dimensionless heat transfer coefficient.

Local Nusselt number: $Nu_x = f(x^*, Re_L, Pr)$

Average Nusselt number: $Nu = f(Re_L, Pr)$

A common form: $Nu = C Re_L^m Pr^n$

Momentum and Heat Transfer - Analogy



When $Pr = 1$, and $\frac{dP^*}{dx^*} = 0$ ($\implies u = u_\infty$ in the free stream, as in flow over a flat plate), then

$$\text{Momentum:} \quad u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{Re_L} \frac{\partial^2 u^*}{\partial y^{*2}}$$

$$\text{Energy:} \quad u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{Re_L} \frac{\partial^2 T^*}{\partial y^{*2}}$$

$$\text{Profile:} \quad u^* = T^*$$

$$\text{Gradients:} \quad \left. \frac{\partial u^*}{\partial y^*} \right|_{y^*=0} = \left. \frac{\partial T^*}{\partial y^*} \right|_{y^*=0}$$

$$\text{Analogy:} \quad \boxed{C_{f,x} \frac{Re_L}{2} = Nu_x}$$

Reynolds analogy

h for fluid with $Pr \approx 1$ from C_f which is easier to measure.



$$\text{St} = \frac{h}{\rho C_p V} = \frac{\text{Nu}}{\text{Re}_L \text{Pr}}$$
$$\boxed{\frac{C_{f,x}}{2} = \text{St}_x} \quad (\text{Pr} = 1)$$

Stanton number is also a dimensionless heat transfer coefficient. It measures the ratio of heat transferred into a fluid to the thermal capacity of fluid.

Reynolds analogy is limited to $\text{Pr} = 1$, and $\frac{dP^*}{dx^*} = 0$.

An analogy that is applicable over a wide range of Pr by adding a Prandtl number correction.



Laminar flow over a flat plate

$$C_{f,x} = 0.664\text{Re}_x^{-1/2}$$

$$\text{Nu} = 0.332\text{Pr}^{1/3}\text{Re}_x^{1/2}$$

$$C_{f,x} \frac{\text{Re}_L}{2} = \text{Nu}_x \text{Pr}^{-1/3}$$

$$\Rightarrow \boxed{\frac{C_{f,x}}{2} = \text{StPr}^{2/3} \equiv j_H}$$

$$0.6 < \text{Pr} < 60$$

Here j_H is called the **Colburn j -factor**.

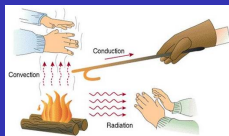
Experiments show that it may be applied for turbulent flow over a surface, even in the presence of pressure gradients ($\frac{dP^*}{dx^*} \neq 0$).

However, the analogy is not applicable for laminar flow unless $\frac{dP^*}{dx^*} = 0$. Therefore, it does not apply to laminar flow in a pipe.



Group	Definition	Interpretation
Biot number (Bi)	$\frac{hL}{k_s}$	Ratio of the internal thermal resistance of a solid to the boundary layer thermal resistance
Bond number (Bo)	$\frac{g(\rho_l - \rho_v)L^2}{\sigma}$	Ratio of gravitational and surface tension forces
Coefficient of friction (C_f)	$\frac{\tau_s}{\rho V^2/2}$	Dimensionless surface shear stress
Eckert number (Ec)	$\frac{V^2}{c_p(T_s - T_\infty)}$	Kinetic energy of the flow relative to the boundary layer enthalpy difference
Fourier number (Fo)	$\frac{\alpha t}{L^2}$	Ratio of the heat conduction rate to the rate of thermal energy storage in a solid. Dimensionless time
Friction factor (f)	$\frac{\Delta p}{(L/D)(\rho u_m^2/2)}$	Dimensionless pressure drop for internal flow
Grashof number (Gr_L)	$\frac{g\beta(T_s - T_\infty)L^3}{\nu^2}$	Measure of the ratio of buoyancy forces to viscous forces
Colburn j factor (j_H)	$St Pr^{2/3}$	Dimensionless heat transfer coefficient
Jakob number (Ja)	$\frac{c_p(T_s - T_{sat})}{h_{fg}}$	Ratio of sensible to latent energy absorbed during liquid-vapor phase change
Mach number (Ma)	$\frac{V}{a}$	Ratio of velocity to speed of sound
Nusselt number (Nu_L)	$\frac{hL}{k_f}$	Ratio of convection to pure conduction heat transfer
Peclet number (Pe_L)	$\frac{VL}{\alpha} = Re_L Pr$	Ratio of advection to conduction heat transfer rates
Prandtl number (Pr)	$\frac{c_p \mu}{k} = \frac{\nu}{\alpha}$	Ratio of the momentum and thermal diffusivities
Reynolds number (Re_L)	$\frac{VL}{\nu}$	Ratio of the inertia and viscous forces
Stanton number	$\frac{h}{\rho V c_p} = \frac{Nu_L}{Re_L Pr}$	Modified Nusselt number

Heat and Mass Transfer



External Flow

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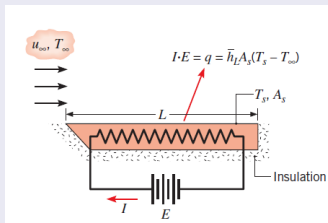
Department of Mechanical Engineering
Indian Institute of Technology Patna

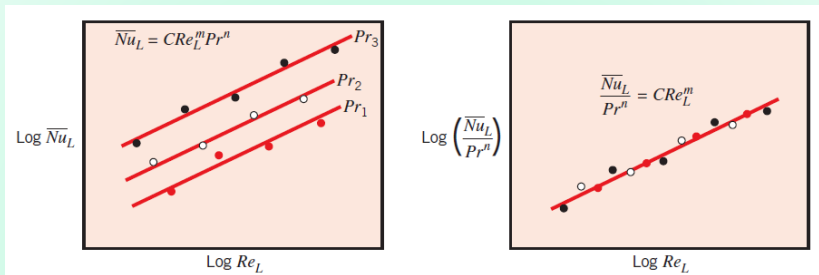
Local Nusselt number: $Nu_x = f(x^*, Re_L, Pr)$

Average Nusselt number: $Nu_L = f(Re_L, Pr)$

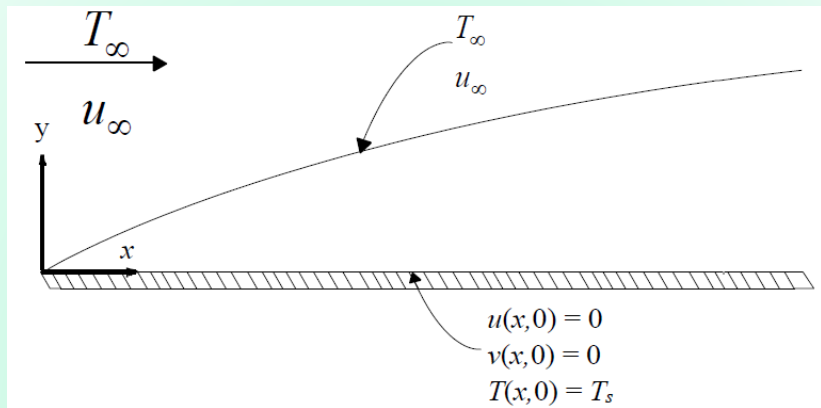
A common form: $Nu_L = C Re_L^m Pr^n$

The Empirical Method





$$T_f \equiv \frac{T_s + T_\infty}{2}$$



Assumptions

Steady, incompressible, laminar flow with constant fluid properties and negligible viscous dissipation.



$$\text{Continuity:} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\text{Momentum:} \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\text{Energy:} \quad u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$

First solved in 1908 by German engineer H. Blasius, a student of L. Prandtl. The profile u/u_∞ remains unchanged with y/δ . A stream function $\psi(x, y)$ is defined as,

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

This takes care of continuity equation.



A dimensionless independent **similarity variable** and a dependent variable such that $u/u_\infty = f'(\eta)$,

$$\eta = y \sqrt{\frac{u_\infty}{\nu x}} \quad \text{and} \quad f(\eta) = \frac{\psi}{u_\infty \sqrt{\nu x / u_\infty}}$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial y} = u_\infty \frac{df}{d\eta} = u_\infty f'$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{1}{2} \sqrt{\frac{\nu u_\infty}{x}} \left(\eta \frac{df}{d\eta} - f \right)$$

$(\because -2x \frac{\partial f}{\partial x} = \eta \frac{\partial f}{\partial \eta})$

$$\boxed{2f''' + ff'' = 0}$$

The problem reduced to one of solving a nonlinear third-order ordinary differential equation.



$$2f''' + ff'' = 0$$

A third-order nonlinear differential equation with boundary conditions:

$$u(x, 0) = v(x, 0) = 0 \text{ and } u(x, \infty) = u_\infty$$

$$\left. \frac{df}{d\eta} \right|_{\eta=0} = f(0) = 0 \text{ and } \left. \frac{df}{d\eta} \right|_{\eta \rightarrow \infty} = 1$$

The problem was first solved by Blasius using a power series expansion approach, and this original solution is known as the [Blasius solution](#).

Similarity function f and its derivatives for laminar boundary layer along a flat plate.

η	f	$\frac{df}{d\eta} = \frac{u}{u_\infty}$	$\frac{d^2f}{d\eta^2}$
0	0	0	0.332
0.5	0.042	0.166	0.331
1.0	0.166	0.330	0.323
1.5	0.370	0.487	0.303
2.0	0.650	0.630	0.267
2.5	0.996	0.751	0.217
3.0	1.397	0.846	0.161
3.5	1.838	0.913	0.108
4.0	2.306	0.956	0.064
4.5	2.790	0.980	0.034
5.0	3.283	0.992	0.016
5.5	3.781	0.997	0.007
6.0	4.280	0.999	0.002
∞	∞	1	0

$$f' = u/u_\infty = 0.99, \text{ for } \eta = 5.0$$

$$y_{\eta=5.0} = \delta = \frac{5.0}{\sqrt{u_\infty/\nu x}} = \frac{5x}{\sqrt{\text{Re}_x}}$$

As $\delta \uparrow$ with $x, \nu \uparrow$ but $\delta \downarrow$ with $u_\infty \uparrow$

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} = \mu u_\infty \sqrt{\frac{u_\infty}{\nu x}} \left. \frac{d^2 f}{d\eta^2} \right|_{\eta=0}$$

$$\Rightarrow \tau_w = 0.332 u_\infty \sqrt{u_\infty/\nu x}$$

$$C_{f,x} = \frac{\tau_w}{\rho u_\infty^2/2} = 0.664 \text{Re}_x^{-1/2}$$

Unlike δ , τ_w and $C_{f,x}$ decrease along the plate as $x^{-1/2}$.



The energy equation:

$$\theta(\eta) = \frac{T(x, y) - T_s}{T_\infty - T_s}$$

$$2 \frac{d^2\theta}{d\eta^2} + \text{Pr} f \frac{d\theta}{d\eta} = 0$$

Boundary conditions:

$$\theta(0) = 0, \theta(\infty) = 1$$

For $\text{Pr} = 1$: δ and δ_t coincide. u/u_∞ and θ are identical for steady, incompressible, laminar flow of a fluid with constant properties over an isothermal flat plate.



$$2 \frac{d^2\theta}{d\eta^2} + \text{Pr} f \frac{d\theta}{d\eta} = 0$$

For $\text{Pr} > 0.6$,

$$\left. \frac{d\theta}{d\eta} \right|_{\eta=0} = 0.332 \text{Pr}^{1/3}$$

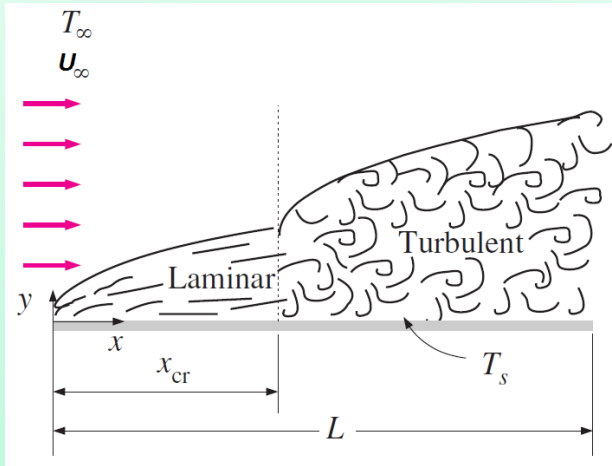
$$\left. \frac{dT}{dy} \right|_{y=0} = 0.332 \text{Pr}^{1/3} (T_\infty - T_s) \sqrt{\frac{u_\infty}{\nu x}}$$

$$h_x = 0.332 \text{Pr}^{1/3} k \sqrt{\frac{u_\infty}{\nu x}}$$

$$\text{Nu}_x = 0.332 \text{Re}_x^{1/2} \text{Pr}^{1/3}$$

$$\delta_t = \frac{\delta}{\text{Pr}^{1/3}} = \frac{5x}{\text{Pr}^{1/3} \sqrt{\text{Re}_x}} \quad (T_f = (T_s + T_\infty)/2)$$

Turbulent Flow



$$Re_x = \frac{\rho u_\infty x}{\mu}$$

$$Re_{cr} = 5 \times 10^5$$



$$\text{Laminar: } \delta_{v,x} = \frac{5x}{\text{Re}_x^{1/2}} \text{ and } C_{f,x} = \frac{0.664}{\text{Re}_x^{1/2}}, \quad \text{Re}_x < 5 \times 10^5$$

$$\text{Turbulent: } \delta_{v,x} = \frac{0.382x}{\text{Re}_x^{1/5}} \text{ and } C_{f,x} = \frac{0.0592}{\text{Re}_x^{1/5}}, \quad 5 \times 10^5 \leq \text{Re}_x \leq 10^7$$

Average skin friction coefficient

$$\text{Laminar: } C_f = \frac{1}{L} \int_0^L C_{f,x} dx = \frac{1.328}{\text{Re}_L^{1/2}}, \quad \text{Re}_x < 5 \times 10^5$$

$$\text{Turbulent: } C_f = \frac{0.074}{\text{Re}_L^{1/5}}, \quad 5 \times 10^5 \leq \text{Re}_x \leq 10^7$$



$$C_f = \frac{1}{L} \left(\int_0^{x_{cr}} C_{f,lam} dx + \int_{x_{cr}}^L C_{f,turb} dx \right)$$

$$C_f = \frac{0.074}{\text{Re}_L^{1/5}} - \frac{1742}{\text{Re}_L}$$

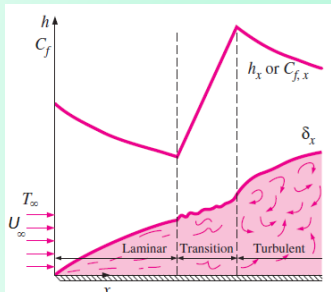
$$(5 \times 10^5 \leq \text{Re}_x \leq 10^7)$$

Rough surface, turbulent: $C_f = \left(1.89 - 1.62 \log \frac{\varepsilon}{L} \right)^{-2.5}$
($\text{Re} > 10^6, \varepsilon/L > 10^{-4}$)

Laminar:
$$Nu_x = \frac{h_x x}{k} = 0.332 Re_x^{0.5} Pr^{1/3} \quad (Pr > 0.6)$$

Turbulent:
$$Nu_x = \frac{h_x x}{k} = 0.0296 Re_x^{0.8} Pr^{1/3} \quad (0.6 \leq Pr \leq 60)$$

$$(5 \times 10^5 \leq Re_x \leq 10^7)$$

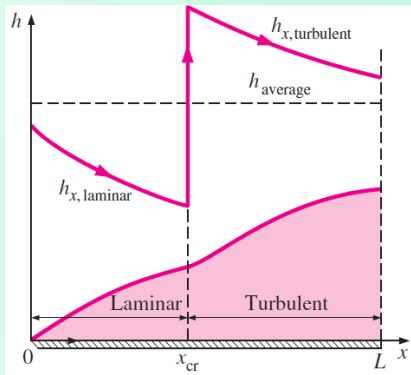


Average values:

$$Nu_{lam} = \frac{hL}{k} = 0.664 Re_x^{0.5} Pr^{1/3}$$

$$Nu_{turb} = \frac{hL}{k} = 0.037 Re_x^{0.8} Pr^{1/3}$$

$$h = \frac{1}{L} \left(\int_0^{x_{cr}} h_{x,lam} dx + \int_{x_{cr}}^L h_{x,turb} dx \right)$$



$$Nu = (0.037Re_L^{0.8} - 871) Pr^{1/3}$$

$(0.6 \leq Pr \leq 60)$

$$(5 \times 10^5 \leq Re_L \leq 10^7)$$



Liquid metals

Such as mercury have high k , very small Pr. Thus, the δ_t develops much faster than δ .

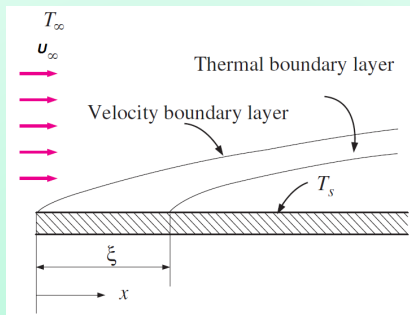
We can assume the velocity in δ_t to be constant at the free stream value and solve the energy equation.

$$\text{Nu}_x = 0.565(\text{Re}_x \text{Pr})^{1/2} = 0.565\text{Pe}_x^{1/2} \quad (\text{Pr} < 0.05, \text{Pe}_x \geq 100)$$

Churchill's correlation for all Prandtl numbers

$$\text{Nu}_x = \frac{0.3387\text{Re}_x^{1/2}\text{Pr}^{1/3}}{\left[1 + (0.0468/\text{Pr})^{2/3}\right]^{1/4}} \quad (\text{Re}_x \text{Pr} \geq 100)$$

Flat Plate with Unheated Starting Length



Laminar:
$$Nu_x = \frac{Nu_x|_{\xi=0}}{[1 - (\xi/x)^{3/4}]^{1/3}} = \frac{0.332 Re_x^{0.5} Pr^{1/3}}{[1 - (\xi/x)^{3/4}]^{1/3}}$$

Turbulent:
$$Nu_x = \frac{Nu_x|_{\xi=0}}{[1 - (\xi/x)^{9/10}]^{1/9}} = \frac{0.0296 Re_x^{0.8} Pr^{1/3}}{[1 - (\xi/x)^{9/10}]^{1/9}}$$



Laminar:
$$\text{Nu}_x = 0.453\text{Re}_x^{0.5} \text{Pr}^{1/3}$$

Turbulent:
$$\text{Nu}_x = 0.0308\text{Re}_x^{0.8} \text{Pr}^{1/3}$$

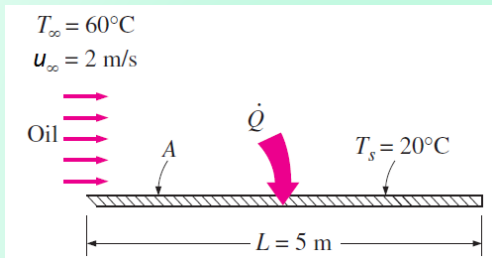
Net heat transfer from the surface:

$$\dot{Q} = q_s A_s$$

$$q_s = h_x [T_s(x) - T_\infty]$$

$$\implies T_s(x) = T_\infty + \frac{q_s}{h_s}$$

Engine oil at 60°C flows over the upper surface of a 5 m long flat plate whose temperature is 20°C with a velocity of 2 m/s. Determine the total drag force and the rate of heat transfer per unit width of the entire plate.



$$\begin{aligned}T_f &= 40^\circ\text{C} \\ \rho &= 876\text{ kg/m}^3 \\ \text{Pr} &= 2870 \\ k &= 0.144\text{ W/m K} \\ \nu &= 242 \times 10^{-6}\text{ m}^2/\text{s}\end{aligned}$$

Known: Engine oil flows over a flat plate.



Find: Total drag force, \dot{Q} per unit width of plate.

Assumptions: The flow is steady, incompressible

$$\text{Re}_L = \frac{u_\infty L}{\nu} = 41322.3 \quad (< \text{Re}_{cr} = 5 \times 10^5)$$

$$C_f = 1.328 \text{Re}_L^{-1/2} = 6.533 \times 10^{-0.3}$$

$$F_D = C_f A_s \frac{\rho u_\infty^2}{2} = 57.23 \text{ N}$$

$$\text{Nu} = 0.664 \text{Re}_L^{1/2} \text{Pr}^{1/3} = 1938.5$$

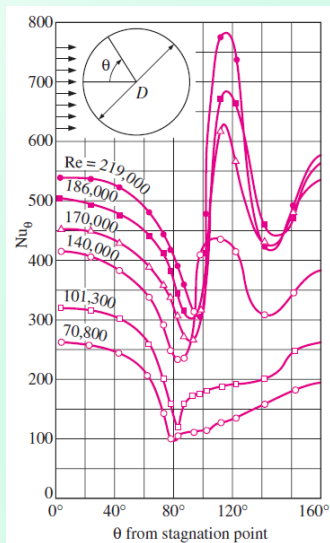
$$h = \frac{k}{L} \text{Nu} = 55.98 \text{ W/m}^2 \text{ K}$$

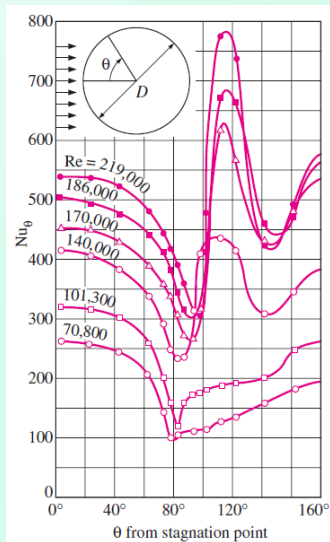
$$\dot{Q} = h A_s (T_\infty - T_s) = 11.2 \text{ W}$$

Churchill and Bernstein correlation:







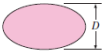
$$Nu_{cyl} = 0.3 + \frac{0.62Re^{1/2}Pr^{1/3}}{[1 + (0.4/Pr)^{2/3}]^{1/4}} \times \left[1 + \left(\frac{Re}{282,000} \right)^{5/8} \right]^{4/5}$$

- Nu is relatively high at the stagnation point. Decreases with increasing θ as a result of thickening of the laminar boundary layer.
- Minimum at 80° , which is the separation point in laminar flow.



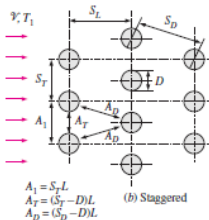
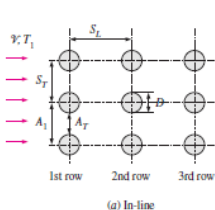
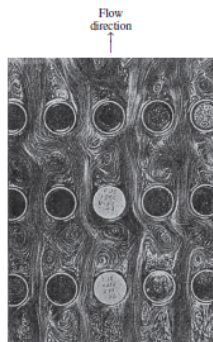


- Increases with increasing Re as a result of the intense mixing in the separated flow region (the wake).
- The sharp increase at about 90° is due to the transition from laminar to turbulent flow.
- The later decrease is again due to the thickening of the boundary layer.
- Nu reaches its second minimum at about 140° , which is the flow separation point in turbulent flow, and increases with Re as a result of the intense mixing in the turbulent wake region.

Cross-section of the cylinder	Fluid	Range of Re	Nusselt number
Circle 	Gas or liquid	0.4–4 4–40 40–4000 4000–40,000 40,000–400,000	$Nu = 0.989Re^{0.330} Pr^{1/3}$ $Nu = 0.911Re^{0.385} Pr^{1/3}$ $Nu = 0.683Re^{0.466} Pr^{1/3}$ $Nu = 0.193Re^{0.618} Pr^{1/3}$ $Nu = 0.027Re^{0.805} Pr^{1/3}$
Square 	Gas	5000–100,000	$Nu = 0.102Re^{0.675} Pr^{1/3}$
Square (tilted 45°) 	Gas	5000–100,000	$Nu = 0.246Re^{0.588} Pr^{1/3}$
Hexagon 	Gas	5000–100,000	$Nu = 0.153Re^{0.638} Pr^{1/3}$
Hexagon (tilted 45°) 	Gas	5000–19,500 19,500–100,000	$Nu = 0.160Re^{0.638} Pr^{1/3}$ $Nu = 0.0385Re^{0.782} Pr^{1/3}$
Vertical plate 	Gas	4000–15,000	$Nu = 0.228Re^{0.731} Pr^{1/3}$
Ellipse 	Gas	2500–15,000	$Nu = 0.248Re^{0.612} Pr^{1/3}$

All properties are evaluated at T_f

Flow Across Tube Banks



S_T Transverse pitch
 S_L Longitudinal pitch
 S_D Diagonal pitch

$$\text{Nu}_D = \frac{hD}{k}$$

$$= C \text{Re}_D^m \text{Pr}^n (\text{Pr}/\text{Pr}_s)^{0.25}$$

Re_D is defined at u_{max}
 but not the u_∞ .

All properties except Pr_s
 are evaluated at
 $(T_{inlet} + T_{outlet})/2$ of fluid.

Pr_s is evaluated at T_s .

Flow Across Tube Banks



Nusselt number correlations for cross flow over tube banks for $N > 16$ and $0.7 < Pr < 500$ (from Zukauskas, Ref. 15, 1987)*

Arrangement	Range of Re_D	Correlation
In-line	0–100	$Nu_D = 0.9 Re_D^{0.4} Pr^{0.36} (Pr/Pr_s)^{0.25}$
	100–1000	$Nu_D = 0.52 Re_D^{0.5} Pr^{0.36} (Pr/Pr_s)^{0.25}$
	1000– 2×10^5	$Nu_D = 0.27 Re_D^{0.63} Pr^{0.36} (Pr/Pr_s)^{0.25}$
	2×10^5 – 2×10^6	$Nu_D = 0.033 Re_D^{0.8} Pr^{0.4} (Pr/Pr_s)^{0.25}$
Staggered	0–500	$Nu_D = 1.04 Re_D^{0.4} Pr^{0.36} (Pr/Pr_s)^{0.25}$
	500–1000	$Nu_D = 0.71 Re_D^{0.5} Pr^{0.36} (Pr/Pr_s)^{0.25}$
	1000– 2×10^5	$Nu_D = 0.35 (S_T/S_L)^{0.2} Re_D^{0.6} Pr^{0.36} (Pr/Pr_s)^{0.25}$
	2×10^5 – 2×10^6	$Nu_D = 0.031 (S_T/S_L)^{0.2} Re_D^{0.8} Pr^{0.36} (Pr/Pr_s)^{0.25}$

*All properties except Pr_s are to be evaluated at the arithmetic mean of the inlet and outlet temperatures of the fluid (Pr_s is to be evaluated at T_s).

$$Nu_{D, N_L < 16} = F Nu_D$$

Correction factor F to be used in $Nu_{D, N_L} = F Nu_D$ for $N_L < 16$ and $Re_D > 1000$ (from Zukauskas, Ref 15, 1987).

N_L	1	2	3	4	5	7	10	13
In-line	0.70	0.80	0.86	0.90	0.93	0.96	0.98	0.99
Staggered	0.64	0.76	0.84	0.89	0.93	0.96	0.98	0.99



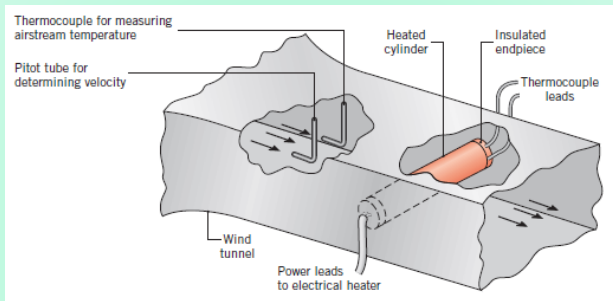
- Become immediately cognizant of the flow geometry.
- Specify the appropriate reference temperature and evaluate the pertinent fluid properties at that temperature.
- Calculate the Reynolds number.
- Decide whether a local or surface average coefficient is required.
- Select the appropriate correlation.

Problem: Cylinder



Experiments have been conducted on a metallic cylinder ($D = 12.7$ mm, $L = 94$ mm). The cylinder is heated internally by an electrical heater and is subjected to a cross flow of air in a low-speed wind tunnel ($V = 10$ m/s, 26.2°C). The heater power dissipation was measured to be $P = 46$ W, while $T_s = 128.4^\circ\text{C}$. It is estimated that 15% of the power dissipation is lost through conduction and radiation.

- 1 Determine h from experimental observations.
- 2 Compare the result with appropriate correlation(s).





Experiments have been conducted on a metallic cylinder ($D = 12.7$ mm, $L = 94$ mm). The cylinder is heated internally by an electrical heater and is subjected to a cross flow of air in a low-speed wind tunnel ($V = 10$ m/s, 26.2°C). The heater power dissipation was measured to be $P = 46$ W, while $T_s = 128.4^\circ\text{C}$. It is estimated that 15% of the power dissipation is lost through conduction and radiation.

$$\text{Nu}_D = 0.26 \text{Re}_D^{0.6} \text{Pr}^{0.37} (\text{Pr}/\text{Pr}_s)^{1/2} \quad (\text{Zhukauskasa relation})$$

Air ($T_\infty = 26.2^\circ\text{C}$):

$$\nu = 15.89 \times 10^{-6} \text{ m}^2/\text{s}, k = 26.3 \times 10^{-3} \text{ W/m K}, \text{Pr} = 0.707$$

Air ($T_f = 77.3^\circ\text{C}$):

$$\nu = 20.92 \times 10^{-6} \text{ m}^2/\text{s}, k = 30 \times 10^{-3} \text{ W/m K}, \text{Pr} = 0.700$$

Air ($T_s = 128.4^\circ\text{C}$): $\text{Pr} = 0.690$

$$\text{Nu}_D = 0.3 + \frac{0.62 \text{Re}_D^{1/2} \text{Pr}^{1/3}}{[1 + (0.4/\text{Pr})^{2/3}]^{1/4}} \left[1 + \left(\frac{\text{Re}_D}{282,000} \right)^{5/8} \right]^{4/5}$$

(Churchill relation)



Experiments have been conducted on a metallic cylinder ($D = 12.7$ mm, $L = 94$ mm). The cylinder is heated internally by an electrical heater and is subjected to a cross flow of air in a low-speed wind tunnel ($V = 10$ m/s, 26.2°C). The heater power dissipation was measured to be $P = 46$ W, while $T_s = 128.4^\circ\text{C}$. It is estimated that 15% of the power dissipation is lost through conduction and radiation.

$$\text{Nu}_D = 0.193 \text{Re}_D^{0.618} \text{Pr}^{1/3} \quad (\text{Hilpert correlation})$$

Air ($T_\infty = 26.2^\circ\text{C}$):

$$\nu = 15.89 \times 10^{-6} \text{ m}^2/\text{s}, k = 26.3 \times 10^{-3} \text{ W/m K}, \text{Pr} = 0.707$$

Air ($T_f = 77.3^\circ\text{C}$):

$$\nu = 20.92 \times 10^{-6} \text{ m}^2/\text{s}, k = 30 \times 10^{-3} \text{ W/m K}, \text{Pr} = 0.700$$

Air ($T_s = 128.4^\circ\text{C}$): $\text{Pr} = 0.690$



The decorative plastic film on a copper sphere of 10 mm diameter is cured in an oven at 75°C. Upon removal from the oven, the sphere is subjected to an airstream at 1 atm and 23°C having a velocity of 10 m/s. Estimate how long it will take to cool the sphere to 35°C.

Copper ($T = 55^\circ\text{C}$):

$$\rho = 8933 \text{ kg/m}^3, k = 399 \text{ W/m K}, C_p = 387 \text{ J/kg}$$

Air ($T_\infty = 23^\circ\text{C}$):

$$\mu = 181.6 \times 10^{-7} \text{ Ns/m}^2, \nu = 15.36 \times 10^{-6} \text{ m}^2/\text{s},$$
$$k = 0.0258 \text{ W/m K}, \text{Pr} = 0.709$$

$$\text{Air } (T_s = 55^\circ\text{C}): \mu = 197.8 \times 10^{-7} \text{ Ns/m}^2$$

$$\text{Nu}_D = 2 + \left(0.4 \text{Re}_D^{1/2} + 0.06 \text{Re}_D^{2/3} \right) \text{Pr}^{0.4} \left(\frac{\mu}{\mu_s} \right)^{1/4}$$

All properties except μ_s are evaluated at T_∞ .

Problem: Sphere

The decorative plastic film on a copper sphere of 10 mm diameter is cured in an oven at 75°C. Upon removal from the oven, the sphere is subjected to an airstream at 1 atm and 23°C having a velocity of 10 m/s. Estimate how long it will take to cool the sphere to 35°C.

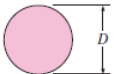
Copper ($T = 55^\circ\text{C}$):

$$\rho = 8933 \text{ kg/m}^3, k = 399 \text{ W/m K}, C_p = 387 \text{ J/kg}$$

Air ($T_\infty = 39^\circ\text{C}$):

$$\nu = 17.15 \times 10^{-6} \text{ m}^2/\text{s}, \text{Pr} = 0.705$$

Air ($T_s = 55^\circ\text{C}$): $\mu = 197.8 \times 10^{-7} \text{ N}\cdot\text{s/m}^2$

Cross-section of the cylinder	Fluid	Range of Re	Nusselt number
Circle 	Gas or liquid	0.4–4 4–40 40–4000 4000–40,000 40,000–400,000	$\text{Nu} = 0.989\text{Re}^{0.330} \text{Pr}^{1/3}$ $\text{Nu} = 0.911\text{Re}^{0.385} \text{Pr}^{1/3}$ $\text{Nu} = 0.683\text{Re}^{0.466} \text{Pr}^{1/3}$ $\text{Nu} = 0.193\text{Re}^{0.618} \text{Pr}^{1/3}$ $\text{Nu} = 0.027\text{Re}^{0.805} \text{Pr}^{1/3}$

All properties are evaluated at T_f



A flat plate of 0.3 m long is maintained at a uniform surface temperature, $T_s = 230^\circ\text{C}$, by using two independently controlled electrical strip heaters. The first heater 0.2 m long and the second one is 0.1 m long. The ambient air temperature at $T_\infty = 25^\circ\text{C}$ flows over the plate at a velocity of 60 m/s. At what heater is the electrical input a maximum? What is the value of this input.

Air ($T_f = 400\text{ K}$):

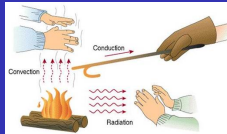
$$\nu = 26.41 \times 10^{-6} \text{ m}^2/\text{s}, k = 0.0338 \text{ W/m K}, \text{Pr} = 0.69$$

Relations

$$\text{Laminar boundary layer} \quad \bar{Nu}_x = \frac{\bar{h}_x x}{k} = 0.664 \text{Re}_x^{1/2} \text{Pr}^{1/3} \quad (\text{Re} < 5 \times 10^5)$$

$$\text{Mixed boundary layer} \quad \bar{Nu}_L = \frac{\bar{h}_L L}{k} = (0.037 \text{Re}_L^{4/5} - 871) \text{Pr}^{1/3} \quad (\text{Re} > 5 \times 10^5)$$

Heat and Mass Transfer



Internal Flow

Sudheer Siddapureddy

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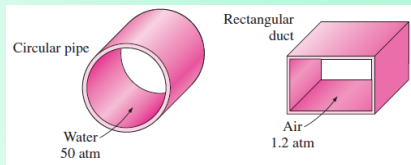
Department of Mechanical Engineering
Indian Institute of Technology Patna

External flow

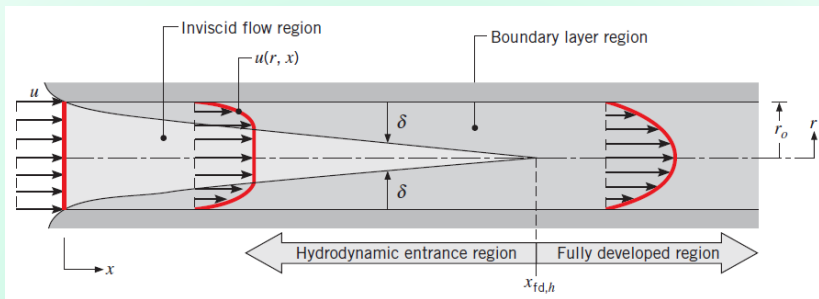
- Fluid has a free surface
- δ is free to grow indefinitely

Internal flow

- Fluid is completely confined by the inner surfaces of the tube
- There is a limit on how much δ can grow



Circular pipes can withstand large pressure difference between inside and outside without distortion. Provide the most heat transfer for the least pressure drop.



$$Re_D = \frac{\rho u_m D}{\mu}$$

Critical

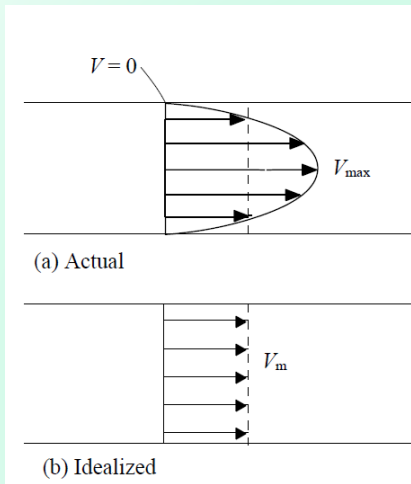
$$Re_{D,c} \approx 2300$$

Laminar

$$\left(\frac{x_{fd,h}}{D}\right)_{\text{lam}} \approx 0.05 Re_D$$

Turbulence

$$\left(\frac{x_{fd,h}}{D}\right)_{\text{turb}} \approx 10$$



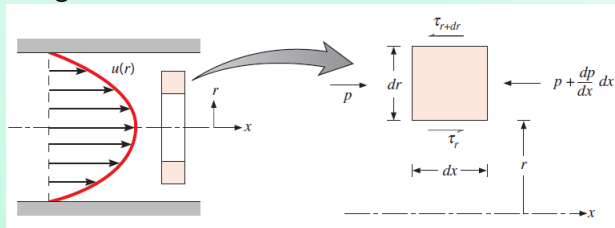
$$\dot{m} = \rho u_m A_c = \int_{A_c} \rho u(r, x) dA_c$$

$$u_m = \frac{2}{r_o^2} \int_0^{r_o} u(r, x) r dr$$

Velocity Profile in Fully Developed Region



Assumptions: Laminar, incompressible, constant property fluid in fully developed region, circular



tube.

$$u(r) = -\frac{1}{4\mu} \left(\frac{dp}{dx} \right) r_o^2 \left[1 - \left(\frac{r}{r_o} \right)^2 \right]$$

$$u_m = -\frac{r_o^2}{8\mu} \frac{dp}{dx}$$

$$\frac{u(r)}{u_m} = 2 \left[1 - \left(\frac{r}{r_o} \right)^2 \right]$$



The Moody (or Darcy) friction factor:

$$f \equiv \frac{-(dp/dx)D}{\rho u_m^2/2}$$

Friction coefficient, Fanning friction factor,

$$C_f = \frac{\tau_s}{\rho u_m^2/2} = \frac{f}{4}$$

For laminar flow:

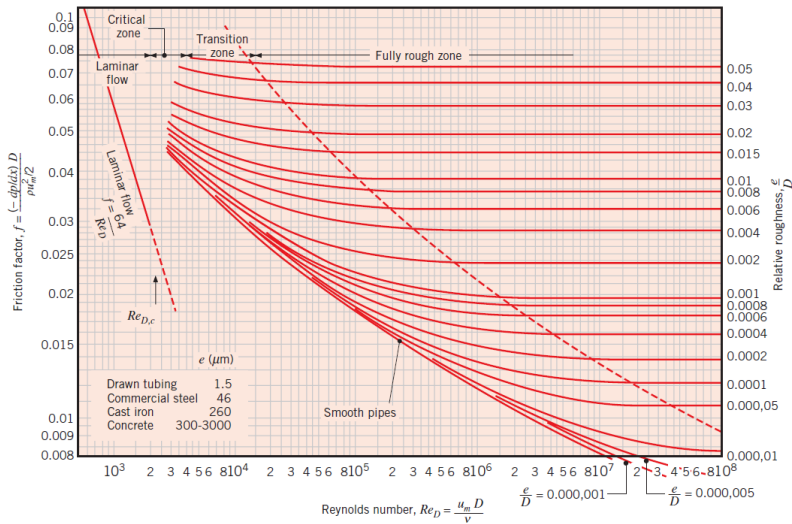
$$f = \frac{64}{\text{Re}}$$

For turbulent flow:

$$f = 0.316 \text{Re}_D^{-1/4} \quad \text{Re}_D \lesssim 2 \times 10^4$$

$$f = 0.184 \text{Re}_D^{-1/5} \quad \text{Re}_D \gtrsim 2 \times 10^4$$

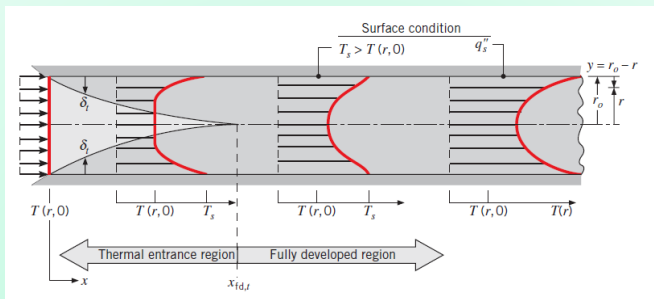
Moody Diagram





$$\Delta p = - \int_{p_1}^{p_2} dp = f \frac{\rho u_m^2}{2D} \int_{x_1}^{x_2} dx = f \frac{\rho u_m^2}{2D} (x_1 - x_2)$$

$$\text{Power, } P = \Delta p \frac{\dot{m}}{\rho}$$



$$Re_D = \frac{\rho u_m D}{\mu}$$

Critical

$$Re_{D,c} \approx 2300$$

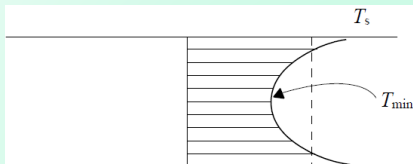
Laminar

$$\left(\frac{x_{fd,t}}{D}\right)_{\text{lam}} \approx 0.05 Re_D Pr$$

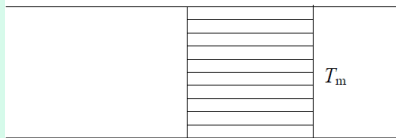
Turbulence

$$\left(\frac{x_{fd,t}}{D}\right)_{\text{turb}} \approx 10$$

Advection rate: integrating the product of mass flux (ρu) and the thermal energy (or enthalpy) per unit mass, $C_p T$, over A_c .



(a) Actual



(b) Idealized

$$\dot{m}C_p T_m = \int_{A_c} \rho u C_p T dA_c$$

For incompressible flow in a circular tube with constant C_p :

$$T_m = \frac{2}{u_m r_o^2} \int_0^{r_o} u T r dr$$

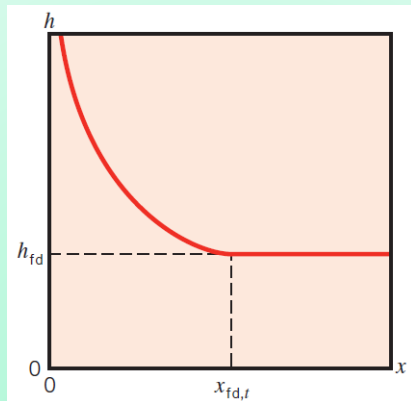
$$q_s'' = h(T_s - T_m)$$

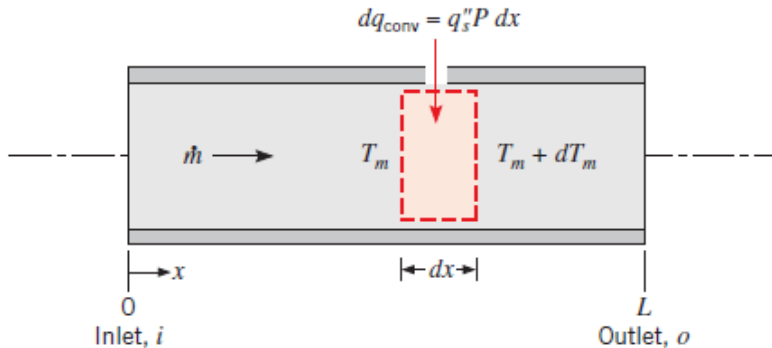


$$\frac{\partial}{\partial x} \left[\frac{T_s(x) - T(r, x)}{T_s(x) - T_m(x)} \right]_{fd,t} = 0$$
$$\Rightarrow \frac{\partial}{\partial x} \left[\frac{T_s(x) - T(r, x)}{T_s(x) - T_m(x)} \right]_{fd,t} = \frac{-\partial T / \partial r|_{r=r_o}}{T_s - T_m} \neq f(x)$$
$$q_s'' = -k \left. \frac{\partial T}{\partial y} \right|_{y=0} = -k \left. \frac{\partial T}{\partial r} \right|_{r=r_o}$$
$$\Rightarrow \frac{h}{k} \neq f(x)$$

In thermally fully developed flow of a fluid with constant properties, h_{local} is a constant, independent of x .

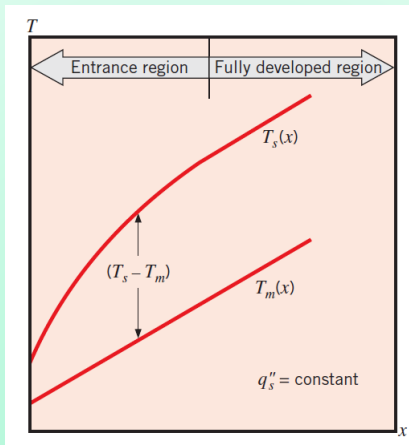
Heat Transfer Coefficient for Flow in a Tube



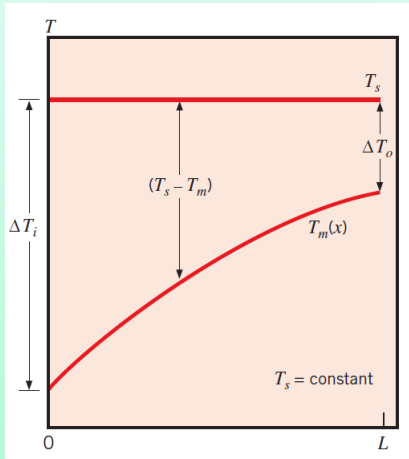


$$dq_{\text{conv}} = \dot{m} C_p [(T_m + dT_m) - T_m]$$

$$dq_{\text{conv}} = \dot{m} C_p dT_m$$

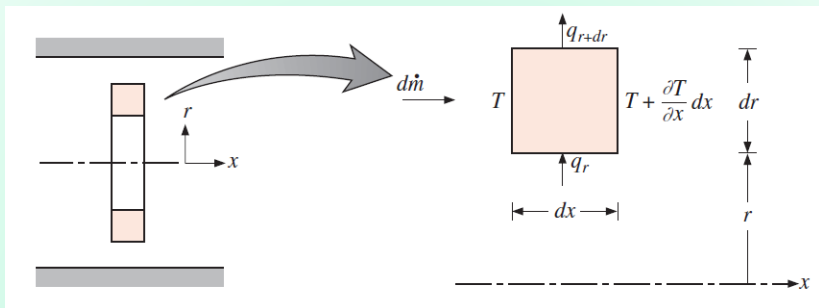


$$T_m(x) = T_{m,i} + \frac{q_s'' P}{\dot{m} C_p} x$$



$$\frac{T_s - T_m(x)}{T_s - T_{m,i}} = \exp\left(-\frac{Px}{\dot{m}C_p \bar{h}}\right)$$

Fully Developed Laminar Flow: Const. q



$$u \frac{\partial T}{\partial x} = \frac{\alpha}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right)$$

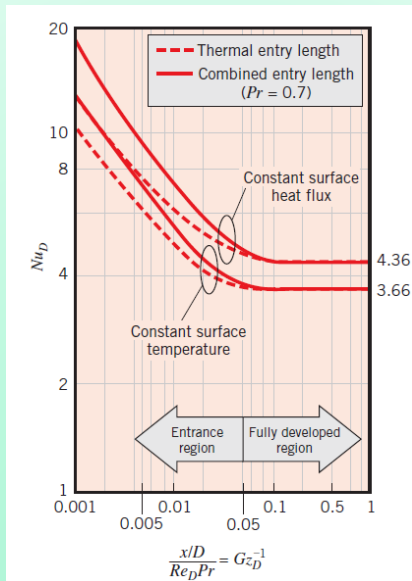
$$T(r, x) \implies T_m(x) - T_s(x) = -\frac{11}{48} \frac{q_s'' D}{k}$$

$$h = \frac{11}{48} \frac{k}{D} \implies \text{Nu}_D = \frac{hD}{k} = 4.36$$



$$\text{Nu}_D = \frac{hD}{k} = 3.66$$

The Entry Region





Dittus-Boelter equation









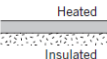

$$\text{Nu}_D = 0.023 \text{Re}_D^{4/5} \text{Pr}^n$$

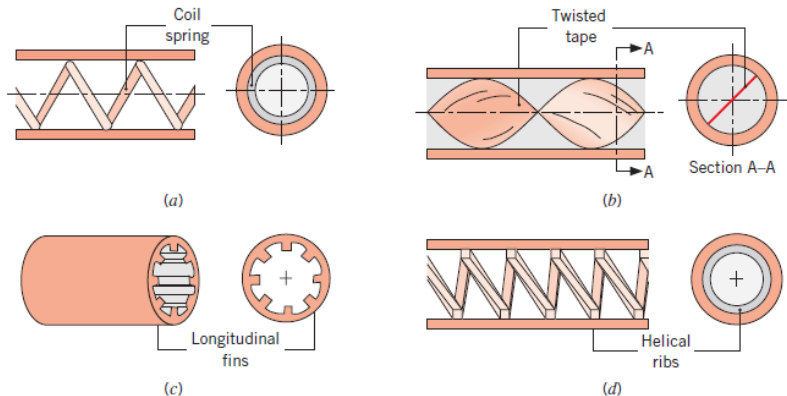
where $n = 0.4$ for heating ($T_s > T_m$) and 0.3 for cooling ($T_s < T_m$).

The Entry Region



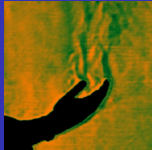
Nusselt numbers and friction factors for fully developed laminar flow in tubes of differing cross section

Cross Section	$\frac{b}{a}$	$Nu_D \equiv \frac{hD_h}{k}$		fRe_{D_h}
		(Uniform q_s'')	(Uniform T_s)	
	—	4.36	3.66	64
	1.0	3.61	2.98	57
	1.43	3.73	3.08	59
	2.0	4.12	3.39	62
	3.0	4.79	3.96	69
	4.0	5.33	4.44	73
	8.0	6.49	5.60	82
	∞	8.23	7.54	96
	∞	5.39	4.86	96
	—	3.11	2.49	53



Internal flow heat transfer enhancement schemes: (a) longitudinal section and end view of coil-spring wire insert, (b) longitudinal section and cross-sectional view of twisted tape insert, (c) cut-away section and end view of longitudinal fins, and (d) longitudinal section and end view of helical ribs.

Heat and Mass Transfer



Free Convection

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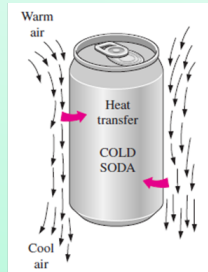
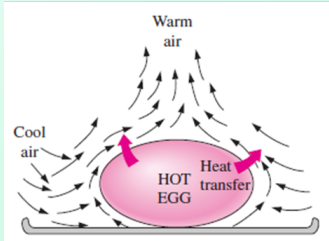
Buoyancy

Combined presence of a fluid density gradient and a body force that is proportional to density.

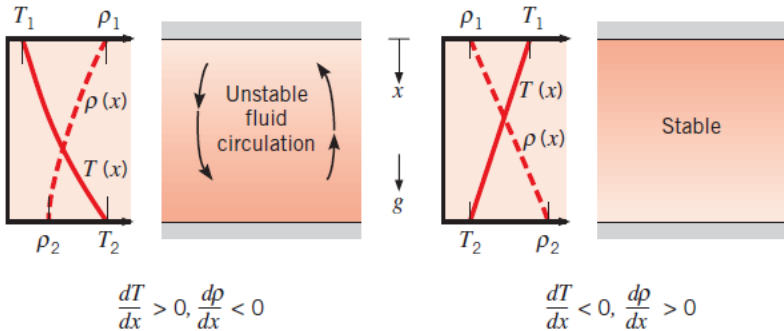
Body Force

- Gravitational
- Centrifugal force in rotating fluid machinery
- Coriolis force in atmospheric
- Oceanic rotational motions

Free Convection: Examples



Conditions of Stable & Unstable T Gradient



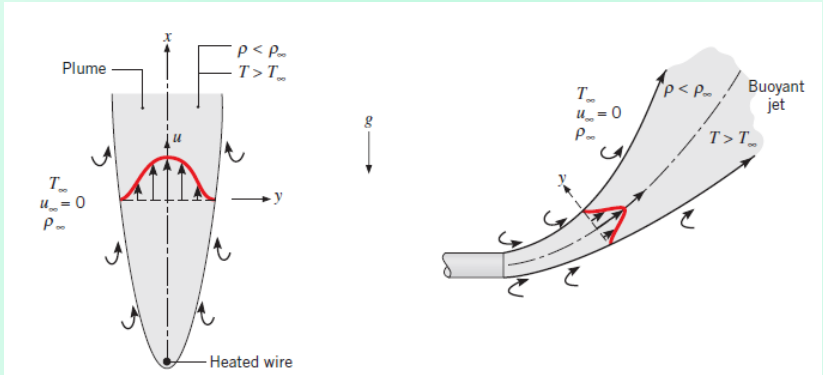
Classification: bounded or unbounded

- Free boundary flows
- Bounded by a surface

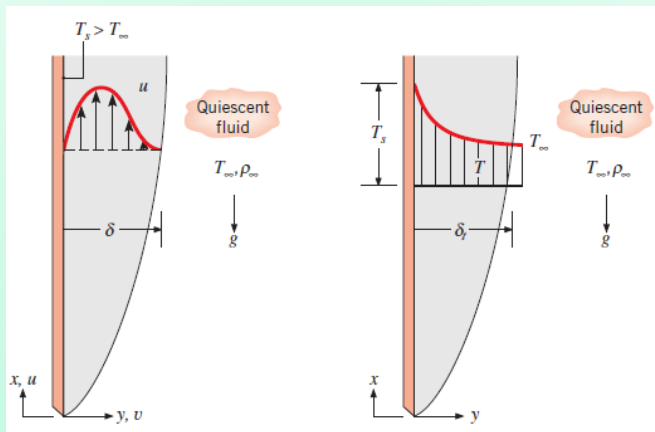
Free Boundary Flows

May occur in the form of a plume or a buoyant jet. plume is associated with fluid rising from a submerged heated object.

A heated wire immersed in an extensive, quiescent fluid.



Heated Vertical Plate



Momentum:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{1}{\rho} X + \nu \frac{\partial^2 u}{\partial y^2}$$

Force by gravity per unit volume:

$$X = -\rho g$$

$$\Rightarrow u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} - g + \nu \frac{\partial^2 u}{\partial y^2}$$

From y-momentum equation: $(\partial p / \partial y) = 0$

$$\frac{\partial P}{\partial x} = -\rho_{\infty} g$$

$$\Rightarrow u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g \frac{\Delta \rho}{\rho} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$\text{where } \Delta \rho = \rho_{\infty} - \rho$$



Thermodynamic property, Volumetric thermal expansion coeff.,

$$\beta = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_p$$

Approximate form:

$$\beta \approx -\frac{1}{\rho} \frac{\Delta \rho}{\Delta T} = -\frac{1}{\rho} \frac{\rho_\infty - \rho}{T_\infty - T}$$

$$(\rho_\infty - \rho) \approx \rho \beta (T - T_\infty)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g \beta (T - T_\infty) + \nu \frac{\partial^2 u}{\partial y^2}$$



Steady incompressible, free convection, laminar flow of a fluid with constant properties:

$$\text{Continuity:} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\text{Momentum:} \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g\beta(T - T_\infty) + \nu \frac{\partial^2 u}{\partial y^2}$$

$$\text{Energy:} \quad u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$

For an ideal gas ($\rho = p/RT$),

$$\beta = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_p = \frac{1}{T}$$



$$x^* \equiv \frac{x}{L}, y^* \equiv \frac{y}{L}, u^* \equiv \frac{u}{u_0}, v^* \equiv \frac{v}{u_0}, T^* \equiv \frac{T - T_\infty}{T_s - T_\infty}$$

u_0 is an arbitrary reference velocity.

$$\text{Momentum: } u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{g\beta(T_s - T_\infty)L}{u_0^2} T^* + \frac{1}{\text{Re}_L} \frac{\partial^2 u^*}{\partial y^{*2}}$$

$$\text{Energy: } u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{\text{Re}_L \text{Pr}} \frac{\partial^2 T^*}{\partial y^{*2}}$$

Coefficient of T^* is in terms of unknown reference velocity u_0 .
Eliminate it to get a dimensionless parameter:

$$\text{Gr}_L \equiv \frac{g\beta(T_s - T_\infty)L}{u_0^2} \left(\frac{u_0 L}{\nu} \right)^2 = \frac{g\beta(T_s - T_\infty)L^3}{\nu^2}$$



$$\text{Gr}_L \equiv \frac{g\beta(T_s - T_\infty)L^3}{\nu^2} = \frac{\text{Buoyancy force}}{\text{Viscous force}}$$

$\frac{\text{Gr}_L}{\text{Re}_L^2} \ll 1 :$ Free convection is neglected

$$\text{Nu}_L = f(\text{Re}_L, \text{Pr})$$

$\frac{\text{Gr}}{\text{Re}^2} \gg 1 :$ Forced convection is neglected

$$\text{Nu}_L = f(\text{Gr}_L, \text{Pr})$$

$\frac{\text{Gr}}{\text{Re}^2} \equiv 1 :$ Combination of free and forced

Boundary conditions:

$$\text{At } y = 0 : \quad u = v = 0, \quad T = T_s$$

$$\text{At } y \rightarrow \infty : \quad u \rightarrow 0, \quad T \rightarrow T_\infty$$

Similarity parameter of the form:

$$\eta \equiv \frac{y}{x} \left(\frac{\text{Gr}_x}{4} \right)^{1/4}$$

and representing the velocity components in terms of a stream function defined as:

$$\psi(x, y) \equiv f(\eta) \left[4\nu \left(\frac{\text{Gr}_x}{4} \right)^{1/4} \right]$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial y} = \frac{2\nu}{x} \text{Gr}_x^{1/2} f'(\eta)$$



$$v = -\frac{\partial\psi}{\partial x} \quad \text{and} \quad T^* \equiv \frac{T - T_\infty}{T_s - T_\infty}$$

The reduced partial differential equations:

$$f''' + 3ff'' - 2(f')^2 + T^* = 0$$

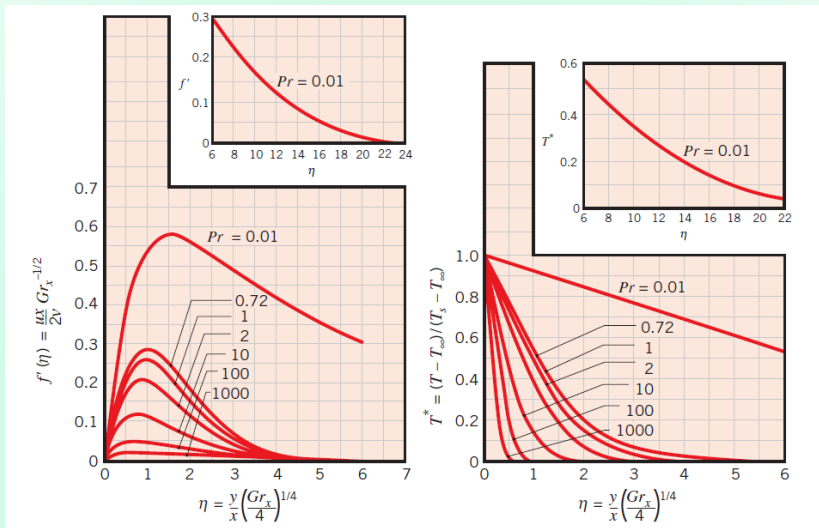
$$T^{*''} + 3PrfT^{*'} = 0$$

Boundary conditions:

$$\text{At } \eta = 0 : \quad f = f' = 0, \quad T^* = 1$$

$$\text{At } \eta \rightarrow \infty : \quad f' \rightarrow 0, \quad T^* \rightarrow 0$$

Laminar, Free Convection



Laminar, free convection boundary layer conditions on a isothermal vertical surface.



Newton's law of cooling:

$$\text{Nu}_x = \frac{hx}{k} = \frac{[q_s''/(T_s - T_\infty)]x}{k}$$

Fourier's law:

$$q_s'' = -k \left. \frac{\partial T}{\partial y} \right|_{y=0} = -\frac{k}{x} (T_s - T_\infty) \left(\frac{\text{Gr}_x}{4} \right)^{1/4} \left. \frac{dT^*}{d\eta} \right|_{\eta=0}$$

$$\Rightarrow \boxed{\text{Nu}_x = \frac{hx}{k} = - \left(\frac{\text{Gr}_x}{4} \right)^{1/4} \left. \frac{dT^*}{d\eta} \right|_{\eta=0} = \left(\frac{\text{Gr}_x}{4} \right)^{1/4} g(\text{Pr})}$$

$$g(\text{Pr}) = \frac{0.75\text{Pr}^{1/2}}{(0.609 + 1.221\text{Pr}^{1/2} + 1.238\text{Pr})^{1/4}}$$

$$0 \leq \text{Pr} \leq \infty$$

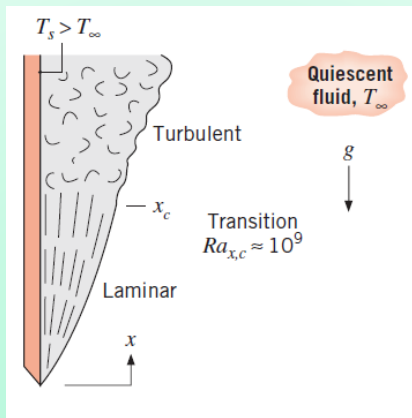


$$\bar{h} = \frac{1}{L} \int_0^L h dx = \frac{k}{L} \left[\frac{g\beta(T_s - T_\infty)}{4\nu^2} \right]^{1/4} g(\text{Pr}) \int_0^L \frac{dx}{x^{1/4}}$$

$$\overline{\text{Nu}}_L = \frac{\bar{h}L}{k} = \frac{4}{3} \left(\frac{\text{Gr}_L}{4} \right)^{1/4} g(\text{Pr})$$

$$\overline{\text{Nu}}_L = \frac{4}{3} \text{Nu}_L$$

These results apply irrespective of whether $T_s > T_\infty$ or $T_s < T_\infty$.



Critical Rayleigh number:

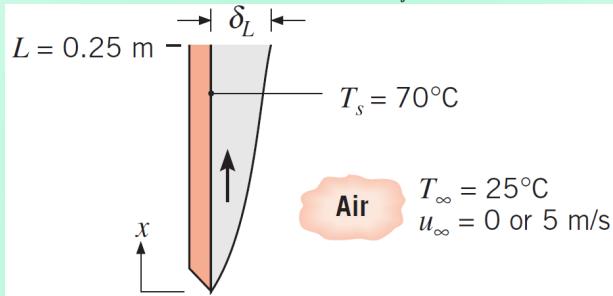
$$Ra_{x,c} = Gr_{x,c} Pr = \frac{g\beta(T_s - T_\infty)x^3}{\nu\alpha} \approx 10^9$$

Problem

Consider a 0.25 m long vertical plate that is at 70°C . The plate is suspended in air that is at 25°C . Estimate the boundary layer thickness at the trailing edge of the plate if the air is quiescent. How does this thickness compare with that which would exist if the air were flowing over the plate at a free stream velocity of 5 m/s?

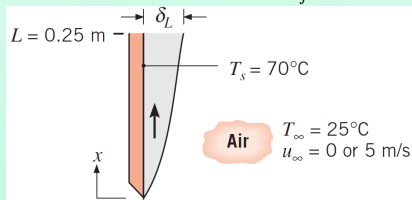
Air properties at $T_f = 47.5^\circ\text{C}$:

$$\nu = 17.95 \times 10^{-6} \text{ m}^2/\text{s}, \text{ Pr} = 0.7, \beta = \frac{1}{T_f} = 3.12 \times 10^{-3} \text{ K}^{-1}$$



Air properties at $T_f = 47.5^\circ\text{C}$:

$$\nu = 17.95 \times 10^{-6} \text{ m}^2/\text{s}, \text{ Pr} = 0.7, \beta = \frac{1}{T_f} = 3.12 \times 10^{-3} \text{ K}^{-1}$$



$$\text{Gr}_L = \frac{g\beta(T_s - T_\infty)L^3}{\nu^2} = 6.69 \times 10^7$$

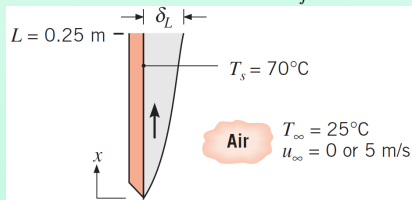
$$\text{Ra}_L = \text{Gr}_L \text{ Pr} = 4.68 \times 10^7$$

For $\text{Pr} = 0.7$, $\eta \approx 6.0$ at the edge of the boundary layer.

$$\eta \equiv \frac{y}{x} \left(\frac{\text{Gr}_x}{4} \right)^{1/4} = 6.0 \implies \delta_L = 0.37 \text{ m}$$

Air properties at $T_f = 47.5^\circ\text{C}$:

$$\nu = 17.95 \times 10^{-6} \text{ m}^2/\text{s}, \text{ Pr} = 0.7, \beta = \frac{1}{T_f} = 3.12 \times 10^{-3} \text{ K}^{-1}$$



For airflow at $u_\infty = 5 \text{ m/s}$

$$\text{Re}_L = \frac{u_\infty L}{\nu} = 6.97 \times 10^4$$

For laminar boundary layer:

$$\delta = \frac{5x}{\sqrt{\text{Re}_L}} = 0.0047 \text{ m} \quad \frac{\delta}{\delta_t} \approx \text{Pr}^{1/3}$$



Comments

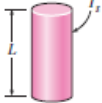
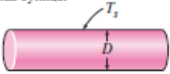
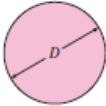
- δ are typically larger for free convection than for forced convection.
- $Gr / Re^2 \ll 1$, and the assumption of negligible buoyancy effects for $u_\infty = 5$ m/s is justified.



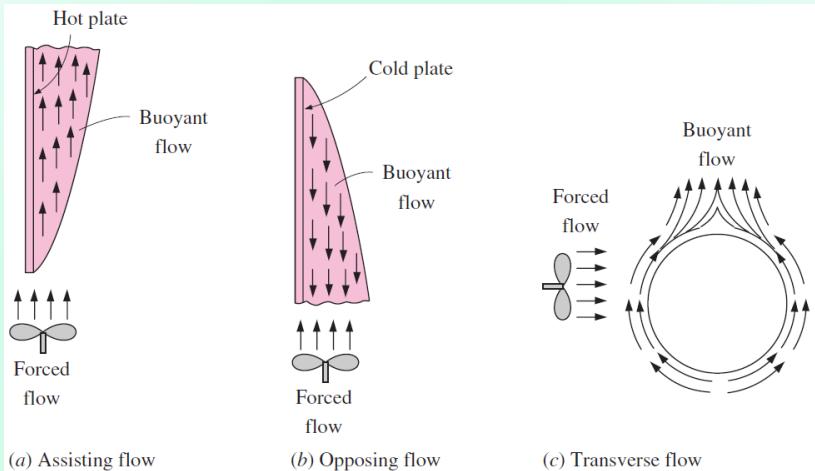
$$\text{Nu} = \frac{h}{L_c} k = C(\text{Gr}_L \text{Pr})^n = C \text{Ra}_L^n$$

$$\text{Ra}_L = \text{Gr}_L \text{Pr} = \frac{g\beta(T_s - T_\infty)L^3}{\nu\alpha}$$

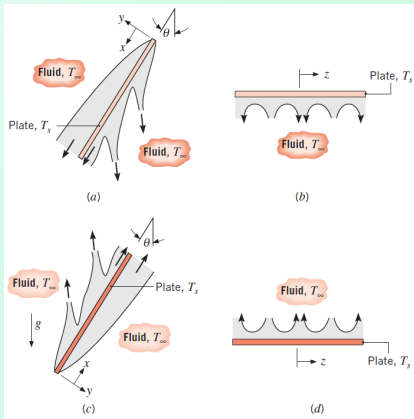
Properties of the fluid are calculated at the mean film temperature,
 $T_f \equiv (T_s + T_\infty)/2$.

Geometry	Characteristic length L_c	Range of Ra	Nu
Vertical cylinder 	L		A vertical cylinder can be treated as a vertical plate when $D \geq \frac{35L}{Gr_D^{1/4}}$
Horizontal cylinder 	D	$Ra_D \leq 10^{12}$	$Nu = \left\{ 0.6 + \frac{0.387Ra_D^{1/4}}{[1 + (0.559/Pr)^{4/5}]^{1/4}} \right\}^2$
Sphere 	D	$Ra_D \leq 10^{11}$ $(Pr \geq 0.7)$	$Nu = 2 + \frac{0.589Ra_D^{1/4}}{[1 + (0.469/Pr)^{4/5}]^{1/4}}$

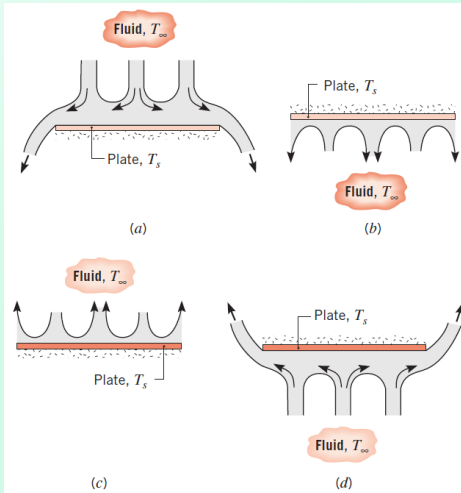
Combined Free and Forced Convection



$$Nu_{\text{combined}}^n = (Nu_{\text{Forced}}^n \pm Nu_{\text{Natural}}^n)$$



Buoyancy-driven flows on an inclined plate: (a) Side view of flows at top and bottom surfaces of a cold plate ($T_s < T_\infty$). (b) End view of flow at bottom surface of cold plate. (c) Side view of flows at top and bottom surfaces of a hot plate ($T_s > T_\infty$). (d) End view of flow at top surface of hot plate.



Buoyancy-driven flows on horizontal cold ($T_s < T_\infty$) and hot ($T_s > T_\infty$) plates: (a) Top surface of cold plate. (b) Bottom surface of cold plate. (c) Top surface of hot plate. (d) Bottom surface of hot plate.

Heat and Mass Transfer



Boiling and Condensation

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Free and Forced convection depends on

$$\rho, C_p, \mu, k_{fluid}$$

Boiling/Condensation Heat Transfer depends on

- $\rho, C_p, \mu, k_{fluid}$
- $\Delta T = |T_s - T_{sat}|$
- Latent heat of vaporization, h_{fg}
- Surface tension at the liquid-vapor interface, σ
- body force arising from the liquid-vapor density difference, $g(\rho_l - \rho_v)$

$$h = h[\Delta T, g(\rho_l - \rho_v), h_{fg}, \sigma, L, \rho, C_p, k, \mu]$$

10 variables in 5 dimensions \implies 5 pi-groups.



$$\frac{hL}{k} = f \left[\frac{\rho g (\rho_l - \rho_v) L^3}{\mu^2}, \frac{C_p \Delta T}{h_{fg}}, \frac{\mu C_p}{k}, \frac{g (\rho_l - \rho_v) L^2}{\sigma} \right]$$

$$\text{Nu}_L = f \left[\frac{\rho g (\rho_l - \rho_v) L^3}{\mu^2}, \text{Ja}, \text{Pr}, \text{Bo} \right]$$

Jakob number

Ratio of max sensible energy absorbed by liquid (vapor) to latent energy absorbed by liquid (vapor) during condensation (boiling).

Bond number

Ratio of the buoyancy force to the surface tension force.

Unnamed parameter

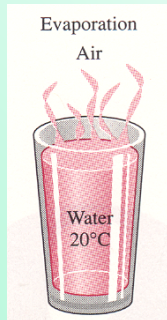
Represents the effect of buoyancy-induced fluid motion on heat transfer.

Boiling

- The process of addition of heat to a liquid such a way that generation of vapor occurs.
- Solid-liquid interface
- Characterized by the rapid formation of vapor bubbles

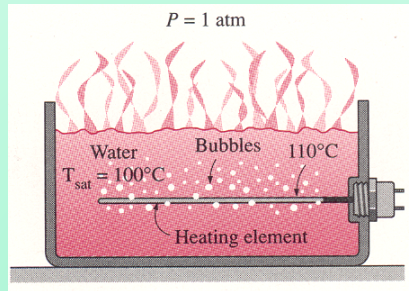
Evaporation

- Liquid-vapor interface
- $P_v < P_{sat}$ of the liquid at a given temp
- No bubble formation or bubble motion



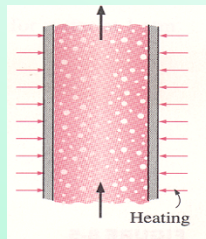
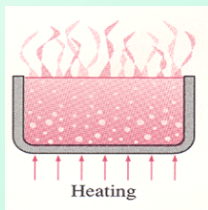
Boiling occurs

- Solid-liquid interface
- when a liquid is brought into contact with a surface at a temperature above the saturation temperature of the liquid





- The boiling processes in practice **do not occur under equilibrium conditions**.
- Bubbles exist because of the surface tension at the liquid vapor interface due to the attraction force on molecules at the interface toward the liquid phase.
- The temperature and pressure of the vapor in a bubble are usually different than those of the liquid.
- Surface tension \downarrow \uparrow Temperature
- Surface tension = 0 at critical temperature
- No bubbles at supercritical pressures and temperatures



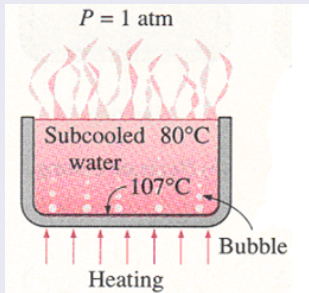
Pool boiling

- Fluid is stationary
- Fluid motion is due to natural convection currents
- Motion of bubbles under the influence of buoyancy

Flow boiling

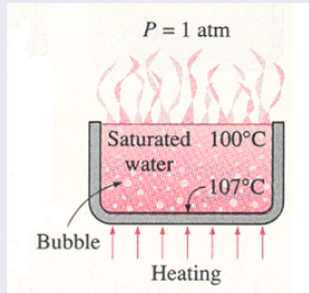
- Fluid is forced to move in a heated pipe or surface by external means such as pump
- Always accompanied by other convection effects

Subcooled boiling



$$T_{\text{bulk of liquid}} < T_{\text{sat}}$$

Saturated boiling

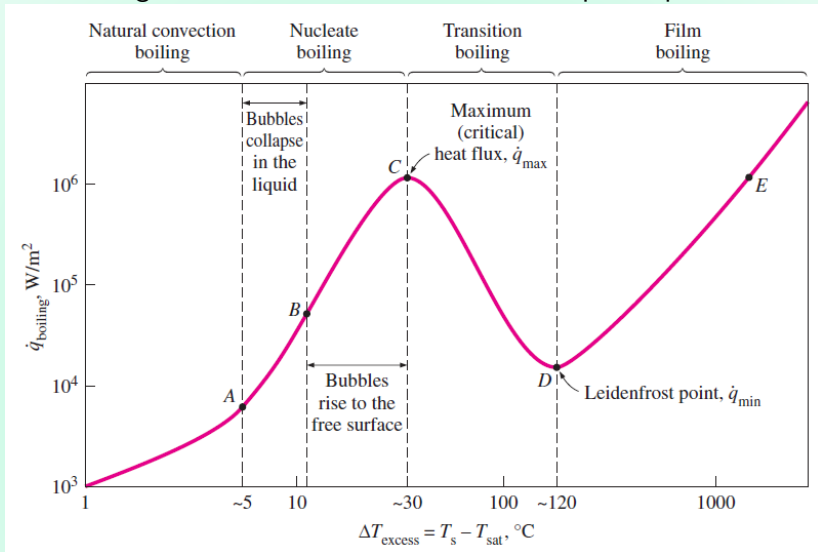


$$T_{\text{bulk of liquid}} = T_{\text{sat}}$$

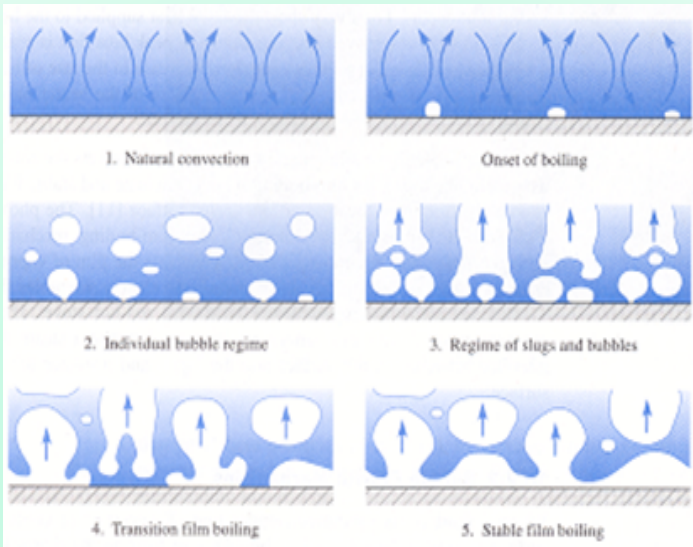
Boiling Regimes - Nukiyama, 1934



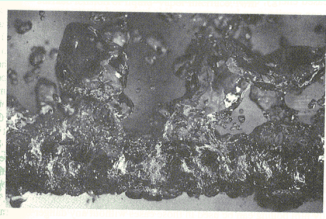
Boiling curve for saturated water at atmospheric pressure



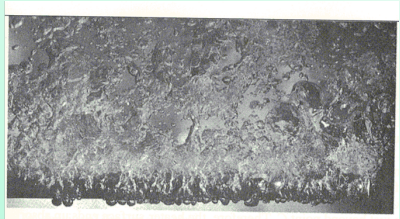
Boiling Regimes - Nukiyama, 1934



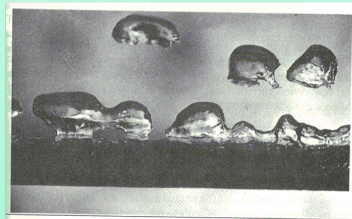
Methanol on horizontal 1 cm steam-heated copper tube



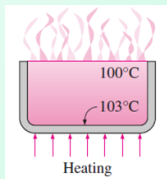
Nucleate boiling



Transition boiling



Film boiling

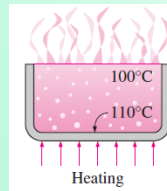


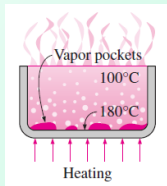
Natural convection

- Governed by natural convection currents
- Heat transfer from the heating surface to the fluid is by natural convection

Nucleate boiling

- Stirring and agitation caused by the entrainment of the liquid to the heater surface increases h, q''
- High heat transfer rates are achieved





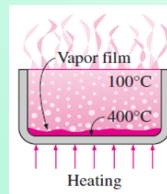
Transition boiling

Unstable film boiling

- Governed by natural convection currents
- Heat transfer from the heating surface to the fluid is by natural convection

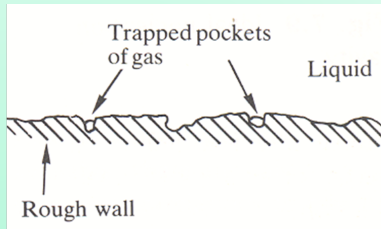
Film boiling

- Presence of vapor film is responsible for the low heat transfer rates
- Heat transfer rate increases with increasing ΔT_e as a result of heat transfer from the heated surface to the liquid through the vapor film by radiation.



The process of bubble formation is called **Nucleation**

Enlarged view of a boiling surface



- The cracks and crevices do not constitute nucleation sites for the bubbles. Must contain pockets of gas/air trapped
- It is from these pockets of trapped air that the vapor bubbles begin to grow during nucleate boiling
- These cavities are the sites at which bubble nucleation occurs



Rohsenow postulated:

- Heat flows from the surface first to the adjacent liquid, as in any single-phase convection process
- High h is a result of local agitation due to liquid flowing behind the wake of departing bubbles

$$q_s'' = \mu_l h_{fg} \left[\frac{g(\rho_l - \rho_v)}{\sigma} \right]^{1/2} \left(\frac{C_{p,l} \Delta T_e}{C_{s,f} h_{fg} Pr_l^n} \right)^3$$

Nucleate boiling

When used to estimate q'' , errors can amount to $\pm 100\%$. The errors for estimating ΔT_e reduce by a factor of 3 $\therefore \Delta T_e \propto (q_s'')^{1/3}$

Heat Transfer in Nucleate Boiling



Surface–Fluid Combination	$C_{s,f}$	n
Water–copper		
Scored	0.0068	1.0
Polished	0.0128	1.0
Water–stainless steel		
Chemically etched	0.0133	1.0
Mechanically polished	0.0132	1.0
Ground and polished	0.0080	1.0
Water–brass	0.0060	1.0
Water–nickel	0.006	1.0
Water–platinum	0.0130	1.0
<i>n</i> -Pentane–copper		
Polished	0.0154	1.7
Lapped	0.0049	1.7
Benzene–chromium	0.0101	1.7
Ethyl alcohol–chromium	0.0027	1.7

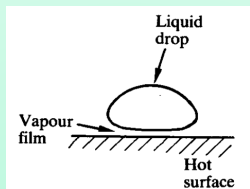


Zuber's correlations for flat horizontal plate:

$$q''_{\max} = 0.149h_{fg}\rho_v \left[\frac{g(\rho_l - \rho_v)}{\sigma} \right]^{1/4}$$

Critical heat flux

Rewetting of hot surfaces: Liquid does not wet hot surface.



$$q''_{\min} = Ch_{fg}\rho_v \left[\frac{g\sigma(\rho_l - \rho_v)}{(\rho_l + \rho_v)^2} \right]^{1/4}$$

C is a non-dimensional constant which lies between 0.09 and 0.18.

$C = 0.09$ provides a better fit.

$C = 0.13$ is sometimes taken as an intermediate value



- Father - Minister
- Started off with Theological studies
- PhD thesis, "On the Harmonious Relationship of Movements in the Human Body"
- Professor at University of Duisburg
- Areas of influences:
 - Theologian
 - Physician (Private Medical practice)
 - As a Prof. taught:
Medicine, Physics, and Chemistry



$$\overline{Nu}_D = \frac{\bar{h}_{\text{conv}} D}{k_v} = C \left[\frac{g(\rho_l - \rho_v) h'_{fg} D^3}{\nu_v k_v (T_s - T_{\text{sat}})} \right]^{1/4}$$

$C = 0.62$ for horizontal cylinders

$C = 0.67$ for spheres

The effective latent heat of vaporization allows for the inclusion of sensible heating effects in the vapor film.

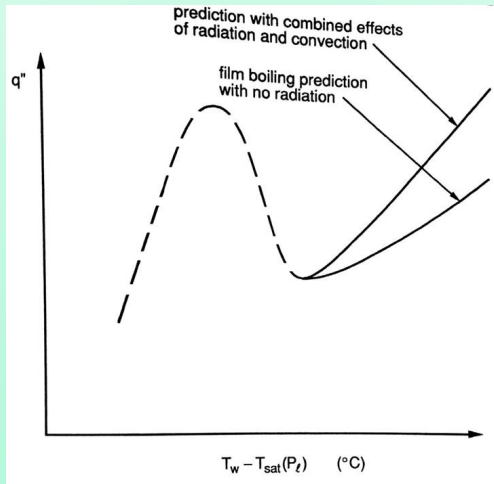
$$h'_{fg} = h_{fg} + 0.5C_{p,v}(T_s - T_{\text{sat}})$$

Vapor properties are evaluated at the film temperature,
 $T_f = (T_s + T_{\text{sat}})/2$.

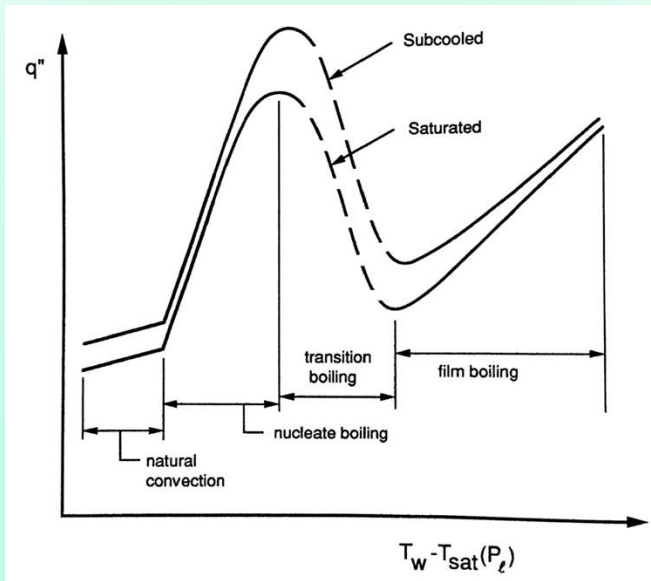
Radiation Effects in Film Boiling



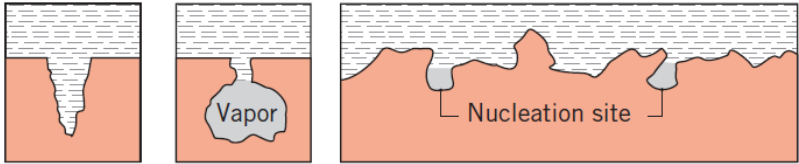
$$h_{\text{total}} = h_{\text{film conv}} + \frac{3}{4}h_{\text{rad}} \quad h_{\text{rad}} = \frac{\epsilon_s \sigma (T_s^4 - T_{\text{sat}}^4)}{T_s - T_{\text{sat}}}$$



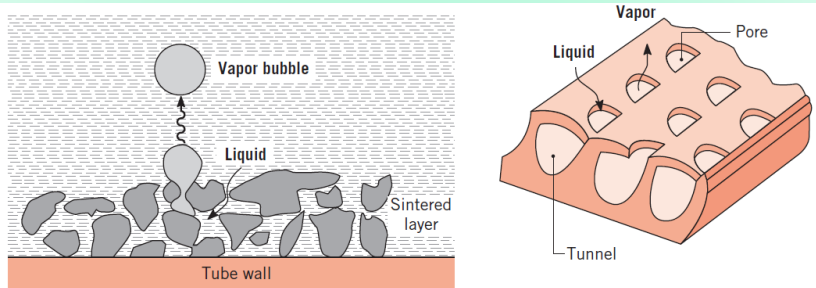
Effect of Liquid Subcooling



Enhancement of Heat Transfer



- Roughening or structuring or coating of the heating surface
- Production of artificial nucleation sites by sintering and
- Addition of gases or liquids or solids





The bottom of a copper pan, 0.3 m in diameter, is maintained at 118°C by an electric heater. Estimate the power required to boil water in this pan. What is the evaporation rate? Estimate the critical heat flux.

Saturated water, liquid at 100°C:

$$\rho_l = 1/v_f = 957.9 \text{ kg/m}^3, C_{p,l} = C_{p,g} = 4.217 \text{ kJ/kg K},$$
$$\mu_l = \mu_f = 279 \times 10^{-6} \text{ N s/m}^2, Pr_l = Pr_f = 1.76,$$
$$h_{fg} = 2257 \text{ kJ/kg}, \sigma = 58.9 \times 10^{-3}$$

Saturated water, vapor at 100°C:

$$\rho_v = 1/v_g = 0.5955 \text{ kg/m}^3$$

Saturated water, liquid at 100°C:

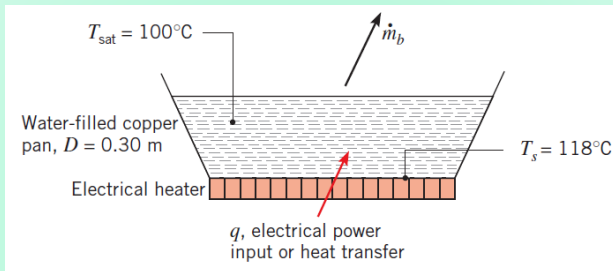
$$\rho_l = 1/v_f = 957.9 \text{ kg/m}^3, C_{p,l} = C_{p,g} = 4.217 \text{ kJ/kg K},$$

$$\mu_l = \mu_f = 279 \times 10^{-6} \text{ N s/m}^2, \text{Pr}_l = \text{Pr}_f = 1.76,$$

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Saturated water, vapor at 100°C:

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Saturated water, liquid at 100°C:

$$\rho_l = 1/v_f = 957.9 \text{ kg/m}^3, C_{p,l} = C_{p,g} = 4.217 \text{ kJ/kg K},$$

$$\mu_l = \mu_f = 279 \times 10^{-6} \text{ N s/m}^2, \text{Pr}_l = \text{Pr}_f = 1.76,$$

$$h_{fg} = 2257 \text{ kJ/kg}, \sigma = 58.9 \times 10^{-3}$$

Saturated water, vapor at 100°C:

$$\rho_v = 1/v_g = 0.5955 \text{ kg/m}^3$$

$$q_s'' = \mu_l h_{fg} \left[\frac{g(\rho_l - \rho_v)}{\sigma} \right]^{1/2} \left(\frac{C_{p,l} \Delta T_e}{C_{s,f} h_{fg} \text{Pr}_l^n} \right)^3$$

$$q_s'' = 55.8 \text{ kW}$$

$$\dot{m}_{\text{evap}} = \frac{q_s}{h_{fg}} 89 \text{ kg/h}$$



Saturated water, liquid at 100°C:

$$\rho_l = 1/v_f = 957.9 \text{ kg/m}^3, C_{p,l} = C_{p,g} = 4.217 \text{ kJ/kg K},$$

$$\mu_l = \mu_f = 279 \times 10^{-6} \text{ N s/m}^2, \text{Pr}_l = \text{Pr}_f = 1.76,$$

$$h_{fg} = 2257 \text{ kJ/kg}, \sigma = 58.9 \times 10^{-3}$$

Saturated water, vapor at 100°C:

$$\rho_v = 1/v_g = 0.5955 \text{ kg/m}^3$$

$$q''_{\max} = 0.149 h_{fg} \rho_v \left[\frac{g(\rho_l - \rho_v)}{\sigma} \right]^{1/4} = 1.26 \text{ MW/m}^2$$

$$q''_{\min} = C h_{fg} \rho_v \left[\frac{g\sigma(\rho_l - \rho_v)}{(\rho_l + \rho_v)^2} \right]^{1/4}$$



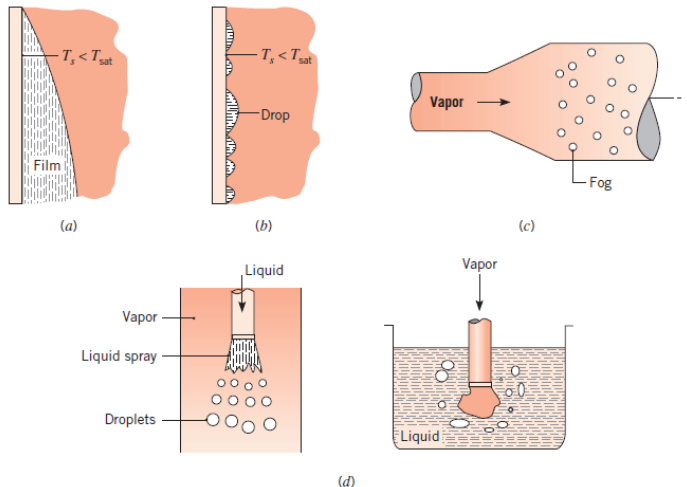
- Condensation is a process in which the removal of heat from a system causes a vapor to convert into liquid.
- Important role in nature:
 - Crucial component of the water cycle
 - Industry
- The spectrum of flow processes associated with condensation on a solid surface is almost a mirror image of those involved in boiling.
- Can also occur on a free surface of a liquid or even in a gas
- Condensation processes are numerous, taking place in a multitude of situations.



- ① Mode of condensation: homogeneous, dropwise, film or direct contact.
- ② Conditions of the vapor: single-component, multicomponent with all components condensable, multicomponent including non-condensable component(s), etc.
- ③ System geometry: plane surface, external, internal, etc.

There are overlaps among different classification methods. Classification based on mode of condensation is the most useful.

Condensation: Classification



Modes of condensation. (a) Film. (b) Dropwise condensation on a surface. (c) Homogeneous condensation or fog formation resulting from increased pressure due to expansion. (d) Direct contact condensation.



- Can happen when vapor is sufficiently cooled $< T_{sat}$ to induce droplet nucleation.
- It may be caused by:
 - Mixing of two vapor streams at different temperatures
 - Radiative cooling of vapor-noncondensable mixtures
 - Fog formation
 - Sudden depressurization of a vapor
 - Cloud formation - adiabatic expansion of warm, humid air masses that rise and cool
 - Cloud - water or ice? -30°C
- Although homogeneous nucleation in pure vapors is possible, in practice dust, other particles act as droplet nucleation embryos



- Droplets form and grow on solid surfaces
- Significant sub-cooling of vapor is required for condensation to start when the surface is smooth and dry.
- The rate of generation of embryo droplets in heterogeneous condensation can be modeled by using kinetic theory
- Heterogeneous condensation leads to:
 - film condensation
 - dropwise condensation



Film condensation

- The surface is blanketed by a liquid film of increasing thickness.
- "Liquid wall" offers resistance.
- Characteristic of clean, uncontaminated surfaces

Dropwise condensation

- Surface is coated with a substance that inhibits wetting
- Drops form in cracks, pits, and cavities.
- Typically, $> 90\%$ of the surface is drops.
- Droplets slide down at a certain size, clearing & exposing surface.
- No resistance to heat transfer in dropwise. h is 10 times higher than in film.

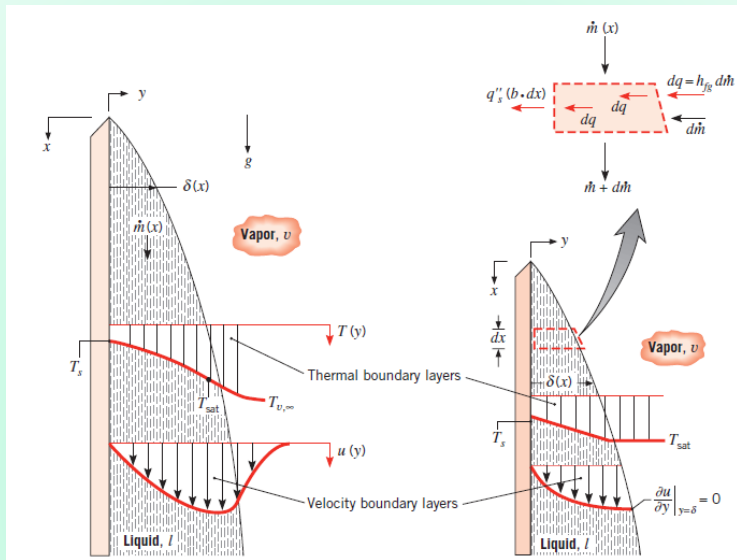


- Providing and maintaining the non-wetting surface characteristics can be difficult.
- The condensate liquid often gradually removes the promoters.
- Furthermore, the accumulation of droplets on a surface can eventually lead to the formation of a liquid film.

It has been postulated that heat transfer occurs at the smaller droplets due to higher thermal resistance in larger drops.

$$\begin{aligned}\bar{h}_{dc} &= 51,104 + 2044T_{sat} & 22^{\circ}\text{C} < T_{sat} < 100^{\circ}\text{C} \\ &= 255,510 & 100^{\circ}\text{C} < T_{sat}\end{aligned}$$

Laminar Film Condensation





Assumptions:

- ① Laminar flow and constant properties
- ② Gas is assumed to be pure vapor and at an uniform T_{sat} .
 - With no temperature gradient in the vapor,
 - Heat transfer to the liquid-vapor interface can occur only by condensation at the interface and not by conduction from the vapor
- ③ Shear stress at the liquid-vapor interface is negligible
 $\partial u / \partial y|_{y=\delta} = 0$
- ④ Momentum and energy transfer by advection in the condensate film are assumed to be negligible.



$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu_l} \frac{dp}{dx} - \frac{X}{\mu_l}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{g}{\mu_l} (\rho_l - \rho_v)$$

$$u(0) = 0, \partial u / \partial y|_{y=\delta} = 0$$

$$u(y) = \frac{g(\rho_l - \rho_v)\delta^2}{\mu_l} \left[\frac{y}{\delta} - \frac{1}{2} \left(\frac{y}{\delta} \right)^2 \right]$$

Mass flow rate per unit width,

$$\Gamma(x) = \frac{\dot{m}(x)}{b} = \int_0^{\delta(x)} \rho_l u(y) dy = \frac{g\rho_l(\rho_l - \rho_v)\delta^3}{3\mu_l}$$



$$dq = h_{fg} d\dot{m}$$

$$dq = q_s''(b dx)$$

$$q_s'' = \frac{k_l(T_{sat} - T_s)}{\delta}$$

$$\frac{d\Gamma}{dx} = \frac{k_l(T_{sat} - T_s)}{h_{fg}}$$

$$\delta(x) = \left[\frac{4k_l\mu_l(T_{sat} - T_S)x}{g\rho_l(\rho_l - \rho_g)h_{fg}} \right]^{1/4}$$



$$\delta(x) = \left[\frac{4k_l \mu_l (T_{sat} - T_s) x}{g \rho_l (\rho_l - \rho_g) h_{fg}} \right]^{1/4}$$

The thermal advection effects may be accounted by:

$$h'_{fg} = h_{fg}(1 + 0.68\text{Ja}) \quad \text{Ja} = \frac{C_p \Delta T}{h_{fg}}$$

$$q''_x = h_x (T_{sat} - T_s)$$

$$h_x = \frac{k_l}{\delta} = \left[\frac{g \rho_l (\rho_l - \rho_v) k_l^3 h'_{fg}}{4 \mu_l (T_{sat} - T_s) x} \right]^{1/4}$$

$$\therefore h_x \propto x^{-1/4}$$

$$\bar{h}_L = \frac{1}{L} \int_0^L h_x dx = \frac{4}{3} h_L = 0.943 \left[\frac{g \rho_l (\rho_l - \rho_v) k_l^3 h'_{fg}}{4 \mu_l (T_{sat} - T_s) L} \right]^{1/4}$$



$$\overline{\text{Nu}}_L = \frac{\bar{h}_L L}{k_l} = \left[\frac{g \rho_l (\rho_l - \rho_v) L^3 h'_{fg}}{4 \mu_l k_l (T_{sat} - T_s)} \right]^{1/4}$$

The total heat transfer to the surface may be obtained by:

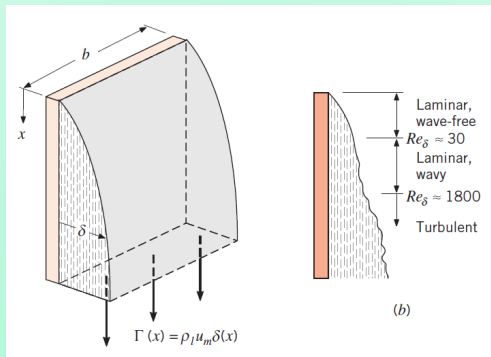
$$q = h_L A (T_{sat} - T_s)$$

$$\dot{m} = \frac{q}{h'_{fg}} = \frac{\bar{h}_L A (T_{sat} - T_s)}{h'_{fg}}$$

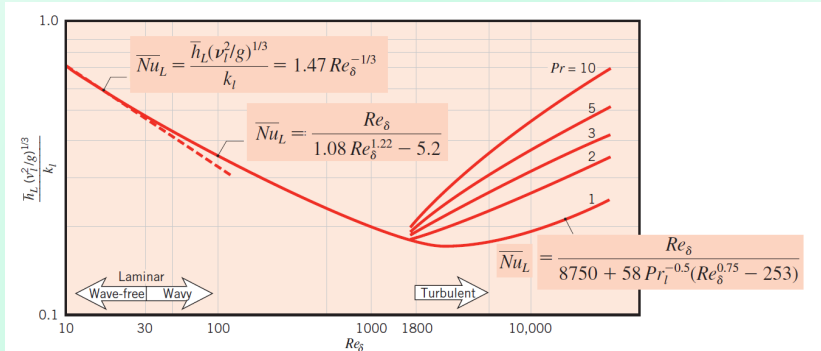
$$Re_{\delta} \equiv \frac{4\Gamma}{\mu_l}$$

Condensate mass flow rate, $\dot{m} = \rho_l u_m b \delta$,

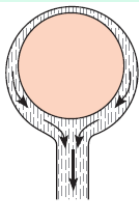
$$Re_{\delta} \equiv \frac{4\dot{m}}{\mu_l b} = \frac{4\rho_l u_m \delta}{\mu_l}$$



Turbulent Film Condensation

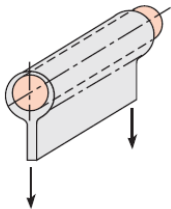


Film Condensation on Radial Systems

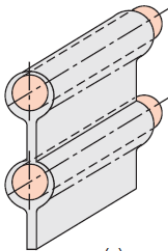


(a)

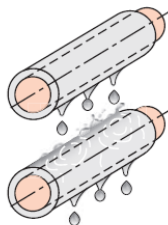
Film condensation on
(a) a sphere, (b) a single horizontal tube,
(c) a vertical tier of horizontal tubes with a
continuous condensate sheet, and (d) with
dripping condensate.



(b)



(c)



(d)



$$\overline{\text{Nu}}_D = \frac{\bar{h}_D D}{k_l} = C \left[\frac{\rho_l g (\rho_l - \rho_v) h'_{fg} D^3}{\mu_l k_l (T_{sat} - T_s)} \right]^{1/4}$$

$C = 0.826$ for sphere and 0.729 for the tube.

For N horizontal unfinned tubes, the average coeff.:

$$\bar{h}_{D,N} = \bar{h}_D N^n$$

$$n = -\frac{1}{4}$$

Heat and Mass Transfer



Heat Exchangers

Sudheer Siddapureddy

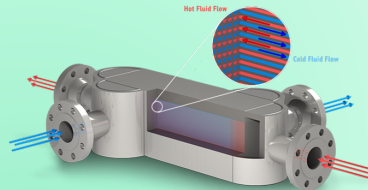
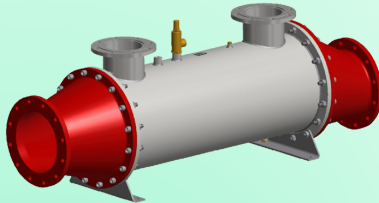
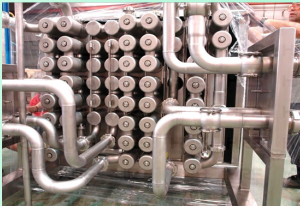
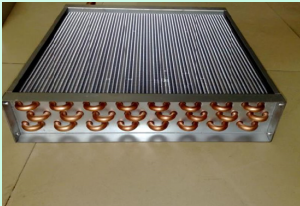
sudheer@iitp.ac.in



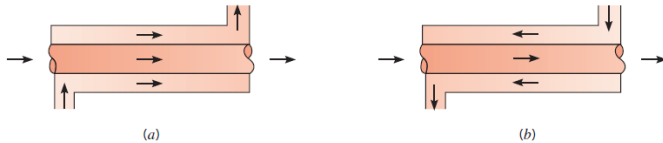
Department of Mechanical Engineering
Indian Institute of Technology Patna

Heat Exchangers

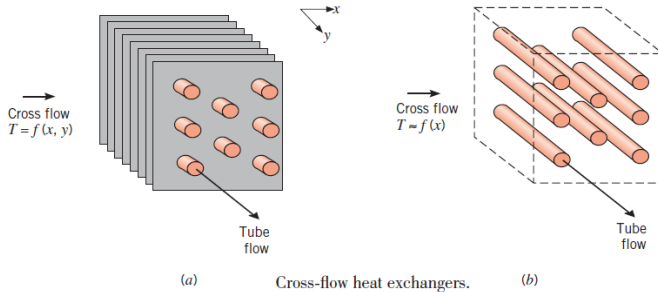
The process of heat exchange between two fluids that are at different temperatures and separated by a solid wall.



Heat Exchanger Types

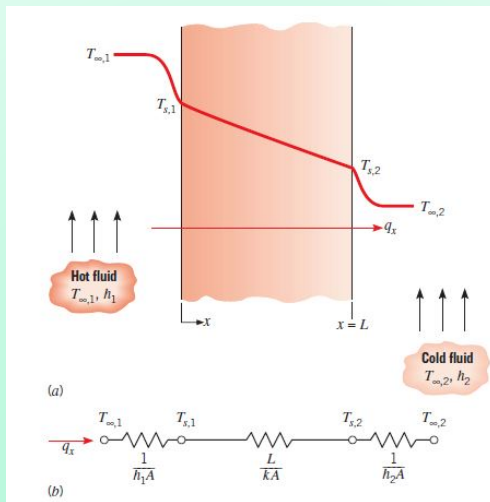


Concentric tube heat exchangers. (a) Parallel flow. (b) Counterflow.



(a) Finned with both fluids unmixed. (b) Unfinned with one fluid mixed and the other unmixed.

Overall Heat Transfer Coefficient



$$\frac{1}{UA} = \frac{1}{U_h A_h} = \frac{1}{U_c A_c} = \frac{1}{h_h A_h} + R_w + \frac{1}{h_c A_c}$$



Fluid	R_f'' ($\text{m}^2 \cdot \text{K}/\text{W}$)
Seawater and treated boiler feedwater (below 50°C)	0.0001
Seawater and treated boiler feedwater (above 50°C)	0.0002
River water (below 50°C)	0.0002–0.001
Fuel oil	0.0009
Refrigerating liquids	0.0002
Steam (nonoil bearing)	0.0001

$$\begin{aligned}\frac{1}{UA} &= \left(\frac{1}{h_h A_h} + \frac{R_{f,h}''}{A_h} \right) + R_w + \left(\frac{1}{h_c A_c} + \frac{R_{f,c}''}{A_c} \right) \\ &= \frac{1}{A_h} \left(\frac{1}{h_h} + R_{f,h}'' \right) + R_w + \frac{1}{A_c} \left(\frac{1}{h_c} + R_{f,c}'' \right)\end{aligned}$$



The overall surface efficiency is expressed as:

$$q = \eta_o h A (T_b - T_\infty)$$

A is the total (fin + exposed base) surface area.

$$\eta_o = 1 - \frac{A_f}{A} (1 - \eta_f)$$

A_f is entire fin surface area, η_f is η of single fin.

For a straight fin with an adiabatic tip:

$$\eta_f = \frac{\tanh(mL)}{mL}$$

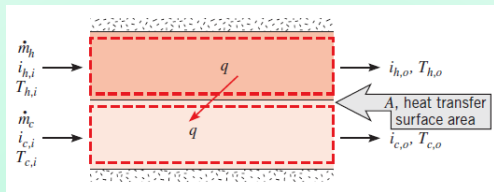
$$m = \sqrt{\frac{hP}{kA_c}}$$



$$\begin{aligned}\frac{1}{UA} &= \frac{1}{\eta_{o,h}A_h} \left(\frac{1}{h_h} + R''_{f,h} \right) + R_w + \frac{1}{\eta_{o,c}A_c} \left(\frac{1}{h_c} + R''_{f,c} \right) \\ &= \frac{1}{(\eta_o U_p A)_h} + R_w + \frac{1}{(\eta_o U_p A)_c}\end{aligned}$$

U_p partial overall heat transfer coefficient.

Fluid Combination	U (W/m ² · K)
Water to water	850–1700
Water to oil	110–350
Steam condenser (water in tubes)	1000–6000
Ammonia condenser (water in tubes)	800–1400
Alcohol condenser (water in tubes)	250–700
Finned-tube heat exchanger (water in tubes, air in cross flow)	25–50



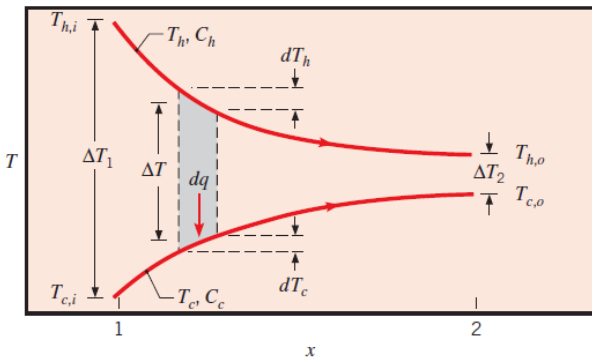
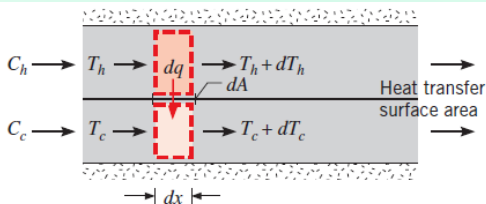
$$q = \dot{m}_h C_{p,h} (T_{h,i} - T_{h,o})$$

$$q = \dot{m}_c C_{p,c} (T_{c,o} - T_{c,i})$$

$$q = U A \Delta T_m$$

$$\Delta T \equiv T_h - T_c$$

Parallel-Flow Heat Exchanger





Assumptions

- ① Insulated from surroundings,
- ② Axial conduction along the tubes is negligible
- ③ PE and KE changes are negligible
- ④ Constant fluid specific heats
- ⑤ U is constant

$$dq = -\dot{m}_h C_{p,h} dT_h \equiv -C_h dT_h$$

$$dq = \dot{m}_c C_{p,c} dT_c \equiv -C_c dT_c$$

C is fluid heat capacity rate.

$$dq = U \Delta T dA$$

$\Delta T = T_h - T_c$ local temp. diff.



$$\begin{aligned}d(\Delta T) &= dT_h - dT_c = -dq \left(\frac{1}{C_h} + \frac{1}{C_c} \right) \\ &= -U \Delta dA \left(\frac{1}{C_h} + \frac{1}{C_c} \right) \\ \int_1^2 \frac{d(\Delta T)}{\Delta T} &= -U \left(\frac{1}{C_h} + \frac{1}{C_c} \right) \int_1^2 dA \\ \ln \left(\frac{\Delta T_2}{\Delta T_1} \right) &= -UA \left(\frac{T_{h,i} - T_{h,o}}{q} + \frac{T_{c,o} - T_{c,i}}{q} \right) \\ &= -\frac{UA}{q} [(T_{h,i} - T_{c,i}) - (T_{h,o} - T_{c,o})] \\ &= -\frac{UA}{q} [\Delta T_2 - \Delta T_1]\end{aligned}$$



$$\begin{aligned}q &= UA \frac{\Delta T_2 - \Delta T_1}{\ln(\Delta T_2 / \Delta T_1)} \\ &= UA \Delta T_{lm}\end{aligned}$$

Log mean temperature difference,

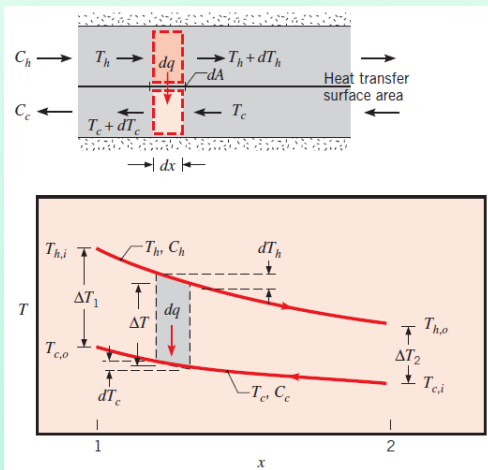
$$\Delta T_{lm} = \frac{\Delta T_2 - \Delta T_1}{\ln(\Delta T_2 / \Delta T_1)} = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1 / \Delta T_2)}$$

For parallel-flow exchanger,

$$\Delta T_1 \equiv T_{h,1} - T_{c,1} = T_{h,i} - T_{c,i}$$

$$\Delta T_2 \equiv T_{h,2} - T_{c,2} = T_{h,o} - T_{c,o}$$

Counter-Flow Heat Exchanger



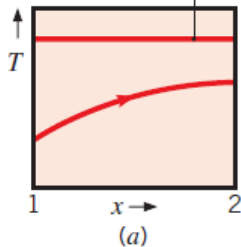
$$\Delta T_1 \equiv T_{h,1} - T_{c,1} = T_{h,i} - T_{c,o}$$

$$\Delta T_2 \equiv T_{h,2} - T_{c,2} = T_{h,o} - T_{c,i}$$

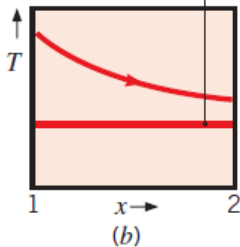
Special Operating Conditions



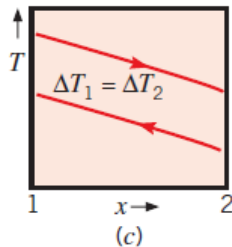
$C_h \gg C_c$
or a condensing
vapor ($C_h \rightarrow \infty$)



$C_h \ll C_c$ or
an evaporating
liquid ($C_c \rightarrow \infty$)



$C_c = C_h$





A counterflow, concentric tube heat exchanger is used to cool the lubricating oil for a large industrial gas turbine engine. The flow rate of cooling water through the inner tube ($D_i = 25$ mm) is 0.2 kg/s, while the flow rate of oil through the outer annulus ($D_o = 45$ mm) is 0.1 kg/s. The oil and water enter at temperatures of 100 and 30°C , respectively. How long must the tube be made if the outlet temperature of the oil is to be 60°C ?

Oil ($\bar{T}_h = 80^\circ\text{C}$):

$$C_p = 2131 \text{ J/kg K}, \mu = 3.25 \times 10^{-2} \text{ N s/m}^2,$$

$k = 0.138 \text{ W/m K}$. Water ($\bar{T}_h = 35^\circ\text{C}$):

$$C_p = 4178 \text{ J/kg K}, \mu = 725 \times 10^{-6} \text{ N s/m}^2, k = 0.625 \text{ W/m K},$$
$$\text{Pr} = 4.85.$$

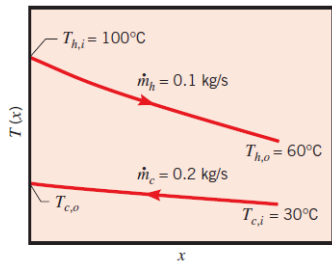
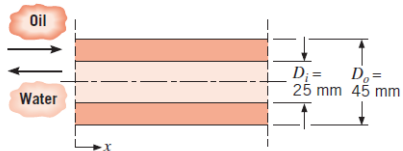
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$C_p = 4178 \text{ J/kg K}$, $\mu = 725 \times 10^{-6} \text{ N s/m}^2$, $k = 0.625 \text{ W/m K}$,

$\text{Pr} = 4.85$.



$$q = \dot{m}_h C_{p,h} (T_{h,i} - T_{h,o}) = \dot{m}_c C_{p,c} (T_{c,o} - T_{c,i})$$

$$q = 8524 \text{ W}, T_{c,o} = 40.2^\circ\text{C}$$

$$q = UA\Delta T_{lm}$$

$$\Delta T_{lm} = 43.2^\circ\text{C} \quad U = \frac{1}{\frac{1}{h_i} + \frac{1}{h_o}}$$

h_i

$$\text{Re}_D = \frac{4\dot{m}_c}{\pi D_i \mu}$$

$$\text{Nu}_D = 0.023 \text{Re}_D^{4/5} \text{Pr}^{0.4} \quad h_i = \text{Nu} \frac{k}{D_i}$$

$$\text{Re}_D = 14050, \text{Nu}_D = 90, h_i = 2250 \text{ W/m}^2 \text{ K}$$

For flow in annulus, $D_h = D_o - D_i = 0.02$ m

$$\text{Re}_D = \frac{\rho u_m D_h}{\mu}$$

$$u_m = \frac{\dot{m}_h}{\rho \pi (D_o^2 - D_i^2) / 4}$$

$$\text{Re}_D = \frac{4\dot{m}_h}{\pi (D_o + D_i) \mu} \quad \text{Nu}_D = \frac{h_o D_h}{k}$$

$$\text{Re}_D = 56, \text{Nu}_D = 5.56, h_o = 38.4 \text{ W/m}^2 \text{ K}$$

$$U = 37.8 \text{ W/m}^2 \text{ K}$$

$$L = \frac{q}{U \pi D_i \Delta T_{\text{lm}}}$$

$$L = 66.5 \text{ m}$$



The max. possible heat transfer rate, q_{max} could, in principle, be achieved in a counterflow heat exchanger of infinite length.

The max. possible temp. diff., $T_{h,i} - T_{c,i}$

$$C_c < C_h : \quad q_{max} = C_c(T_{h,i} - T_{c,i})$$

$$C_h < C_c : \quad q_{max} = C_h(T_{h,i} - T_{c,i})$$

$$q_{max} = C_{min}(T_{h,i} - T_{c,i})$$

Effectiveness,

$$\varepsilon = \frac{q}{q_{max}}$$

$$\varepsilon = \frac{C_h(T_{h,i} - T_{h,o})}{C_{min}(T_{h,i} - T_{c,i})} = \frac{C_c(T_{c,o} - T_{c,i})}{C_{min}(T_{h,i} - T_{c,i})}$$



$$q = \varepsilon C_{min}(T_{h,i} - T_{c,i})$$

Typically,

$$\varepsilon = f\left(\text{NTU}, \frac{C_{min}}{C_{max}}\right)$$

The number of transfer units (NTU) is a dimensionless parameter,

$$\text{NTU} = \frac{UA}{C_{min}}$$

NTU is defined as the ratio of the system's ability to transfer heat (UA) to the fluid's minimum ability to absorb heat (C_{min}).

NTU indicates the size of the heat exchanger required for a given U and C_{min} .



Consider a parallel-flow with $C_{min} = C_h$.

$$\varepsilon = \frac{T_{h,i} - T_{h,o}}{T_{h,i} - T_{c,i}}$$

$$C_r = \frac{C_{min}}{C_{max}} = \frac{T_{c,o} - T_{c,i}}{T_{h,i} - T_{h,o}}$$

$$\varepsilon = \frac{1 - e^{-NTU(1+C_r)}}{1 + C_r}$$

Heat Exchanger Effectiveness Relations



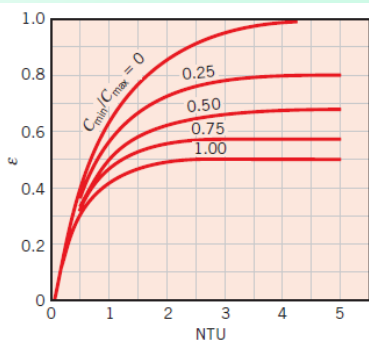
Flow Arrangement	Relation
Parallel flow	$\varepsilon = \frac{1 - \exp[-NTU(1 + C_r)]}{1 + C_r}$
Counterflow	$\varepsilon = \frac{1 - \exp[-NTU(1 - C_r)]}{1 - C_r \exp[-NTU(1 - C_r)]} \quad (C_r < 1)$
	$\varepsilon = \frac{NTU}{1 + NTU} \quad (C_r = 1)$
Shell-and-tube	
One shell pass (2, 4, ... tube passes)	$\varepsilon_1 = 2 \left\{ 1 + C_r + (1 + C_r^2)^{1/2} \times \frac{1 + \exp[-(NTU)_1(1 + C_r^2)^{1/2}]}{1 - \exp[-(NTU)_1(1 + C_r^2)^{1/2}]} \right\}^{-1}$
n shell passes ($2n, 4n, \dots$ tube passes)	$\varepsilon = \left[\left(\frac{1 - \varepsilon_1 C_r}{1 - \varepsilon_1} \right)^n - 1 \right] \left[\left(\frac{1 - \varepsilon_1 C_r}{1 - \varepsilon_1} \right)^n - C_r \right]^{-1}$
Cross-flow (single pass)	
Both fluids unmixed	$\varepsilon = 1 - \exp \left[\left(\frac{1}{C_r} \right) (NTU)^{0.22} \{ \exp[-C_r(NTU)^{0.78}] - 1 \} \right]$
C_{\max} (mixed), C_{\min} (unmixed)	$\varepsilon = \left(\frac{1}{C_r} \right) (1 - \exp\{-C_r[1 - \exp(-NTU)]\})$
C_{\min} (mixed), C_{\max} (unmixed)	$\varepsilon = 1 - \exp(-C_r^{-1}\{1 - \exp[-C_r(NTU)]\})$
All exchangers ($C_r = 0$)	$\varepsilon = 1 - \exp(-NTU)$

Heat Exchanger NTU Relations

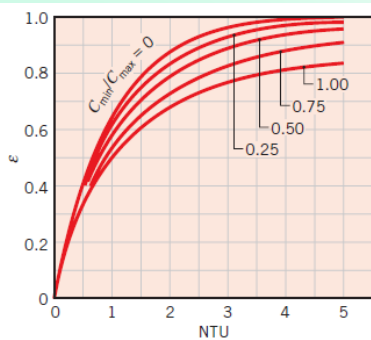


Flow Arrangement	Relation
Parallel flow	$NTU = -\frac{\ln[1 - \varepsilon(1 + C_r)]}{1 + C_r}$
Counterflow	$NTU = \frac{1}{C_r - 1} \ln\left(\frac{\varepsilon - 1}{\varepsilon C_r - 1}\right) \quad (C_r < 1)$
	$NTU = \frac{\varepsilon}{1 - \varepsilon} \quad (C_r = 1)$
Shell-and-tube	
One shell pass (2, 4, ... tube passes)	$(NTU)_1 = -(1 + C_r^2)^{-1/2} \ln\left(\frac{E - 1}{E + 1}\right)$ $E = \frac{2\varepsilon_1 - (1 + C_r)}{(1 + C_r^2)^{1/2}}$
n shell passes ($2n, 4n, \dots$ tube passes)	Use Equations 11.30b and 11.30c with $\varepsilon_1 = \frac{F - 1}{F - C_r} \quad F = \left(\frac{\varepsilon C_r - 1}{\varepsilon - 1}\right)^{1/n} \quad NTU = n(NTU)_1$
Cross-flow (single pass)	
C_{\max} (mixed), C_{\min} (unmixed)	$NTU = -\ln\left[1 + \left(\frac{1}{C_r}\right) \ln(1 - \varepsilon C_r)\right]$
C_{\min} (mixed), C_{\max} (unmixed)	$NTU = -\left(\frac{1}{C_r}\right) \ln[C_r \ln(1 - \varepsilon) + 1]$
All exchangers ($C_r = 0$)	$NTU = -\ln(1 - \varepsilon)$

Heat Exchanger NTU Relations

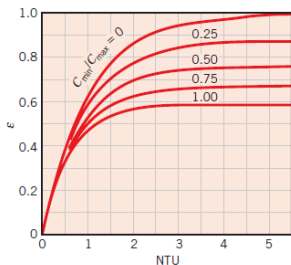
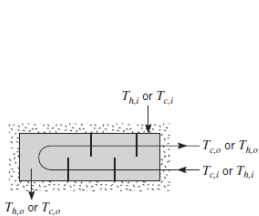


Effectiveness of a parallel- flow heat exchanger

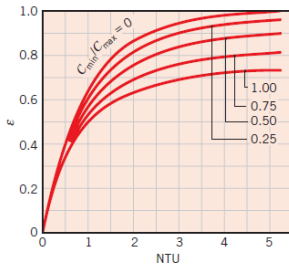
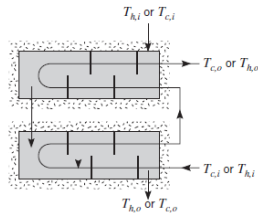


Effectiveness of a counterflow heat exchanger

Heat Exchanger NTU Relations

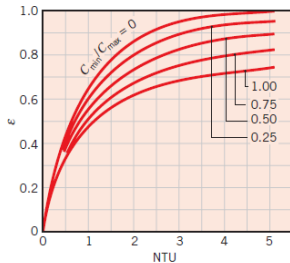
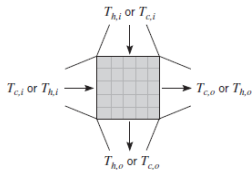


Effectiveness of a shell-and-tube heat exchanger with one shell and any multiple of two tube passes (two, four, etc., tube passes)

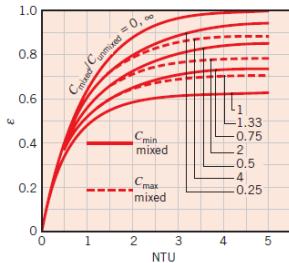
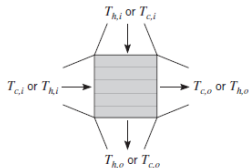


Effectiveness of a shell-and-tube heat exchanger with two shell passes and any multiple of four tube passes (four, eight, etc., tube passes) ($n = 2$).

Heat Exchanger NTU Relations



Effectiveness of a single-pass, cross-flow heat exchanger with both fluids unmixed



Effectiveness of a single-pass, cross-flow heat exchanger with one fluid mixed and the other unmixed



Prescribed

- ① Fluid inlet temperatures
- ② Flow rates
- ③ Desired hot or cold fluid outlet temperatures

Problem

- Specify a specific heat exchanger type
- Determine size, A to achieve the desired outlet temp.
- Custom-built



An **existing heat exchanger** is analyzed to determine

- ① Heat transfer rate
- ② Fluid outlet temperatures for prescribed flow rates
- ③ Inlet temperatures

Associated with

- Off-the-shelf heat exchanger types and sizes available from a vendor



For **design problems**

- 1 Calculate ε and C_r
- 2 Obtain NTU from appropriate eq. or chart
- 3 Determine A

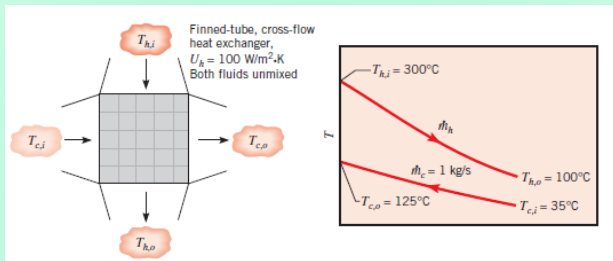
For **performance calculations**

- 1 Calculate NTU and C_r
- 2 Obtain ε from appropriate eq. or chart
- 3 $q_{max} = C_{min}(T_{h,i} - T_{c,i})$
- 4 $q = \varepsilon q_{max}$

Hot exhaust gases, which enter a finned-tube, cross-flow heat exchanger at 300°C and leave at 100°C , are used to heat pressurized water at a flow rate of 1 kg/s from 35 to 125°C . The exhaust gas specific heat is approximately 1000 J/kg K , and the overall heat transfer coefficient based on the gas-side surface area is $U_h = 100\text{ W/m}^2\text{ K}$. Determine the required gas-side surface area A_h using the NTU method.

Water ($\bar{T}_c = 80^{\circ}\text{C}$): $C_{p,c} = 4197\text{ J/kg K}$

$$A_h = 40\text{ m}^2.$$



Heat and Mass Transfer



Radiation

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Indian Institute of Technology Patna

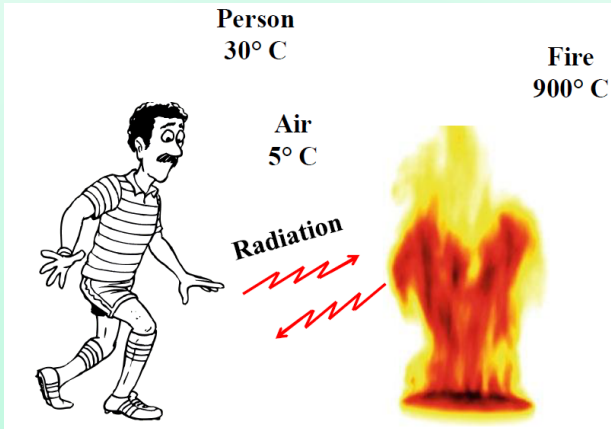


- Thermal radiation requires no matter
- Applications: Industrial heating, cooling and drying processes, energy conservation methods - fossil fuel combustion and solar radiation

Objectives

- Means by which thermal radiation is generated
- Specific nature of radiation
- Manner in which radiation interacts with matter

Heat Exchanger Types

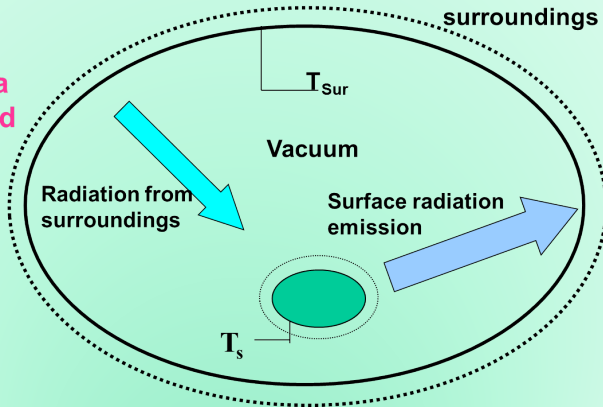


Radiation can occur between two bodies, even when they are separated by a medium colder than both.

Solar radiation reaches the earth after passing through cold air layers at high altitudes.

- $T_s > T_{sur}$
- No conduction or convection - still solid will cool
- Solid gets cooled - emission of thermal radiation from the surface of the solid

**Radiation
cooling of a
heated solid**





Radiation - Propagation of electromagnetic waves

J.C. Maxwell - accelerated charges or changing electric currents give rise to electric and magnetic fields. These moving fields are called **Electromagnetic Waves or Electromagnetic Radiation**

Electro-magnetic Radiation - energy emitted by matter as a result of the changes in the electronic configurations of the atoms or molecules.



Characteristics of Electro-magnetic Radiation

- Frequency ν (Hz - 1/s)
- Wavelength λ (m)

$$\lambda = \frac{c}{\nu} \qquad c = \frac{c_o}{n}$$

In vacuum $c_o = 2.998 \times 10^8$ m/s and n is the index of refraction

Material	n
Air and most gases	1.0
Glass	1.5
Water	1.33

Observations

λ, c : depend on medium through which wave travels.

ν : independent of the medium depends only on the source.



Electro-magnetic Radiation Propagation of a discrete packets of energy called photons or quanta

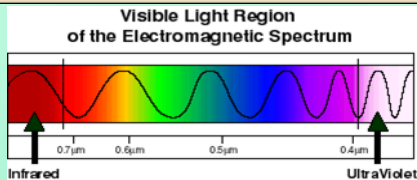
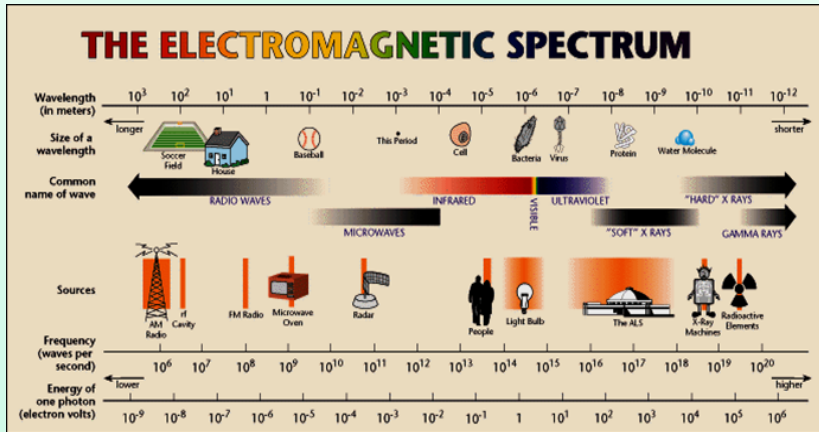
Each photon of frequency ν is considered to have an energy of

$$e = h\nu = \frac{hc}{\lambda}$$

Planck's constant, $h = 6.625 \times 10^{-34}$ J s

- Energy of the photon is inversely proportional to its wavelength
- Shorter wavelength radiation possess larger photon energies
- No wonder, we try to avoid very short wavelength radiation such as gamma rays and X-rays since they are highly destructive

Electromagnetic Spectrum





Color	Wavelength band
Violet	0.40-0.44 μm
Blue	0.44-0.49 μm
Green	0.49-0.54 μm
Yellow	0.54-0.60 μm
Orange	0.60-0.67 μm
Red	0.63-0.76 μm

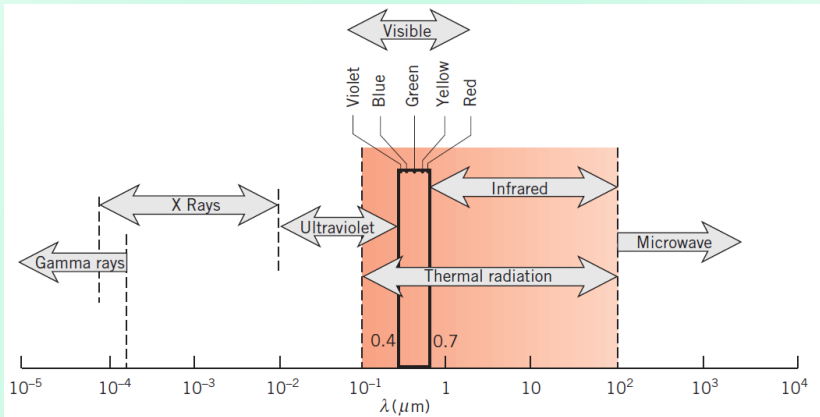
Solar radiation

- Electromagnetic radiation emitted by sun, 0.3 - 3 μm
- Some portion in visible range
- Other portion in ultraviolet and infrared range

Thermal Radiation

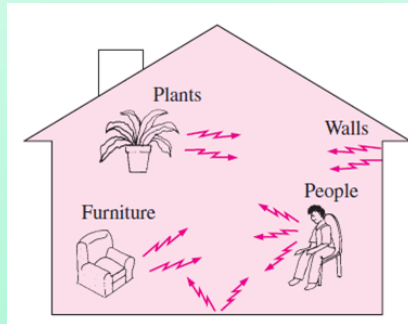


In heat transfer, we are interested in energy emitted by bodies due to their temperature only, **Thermal radiation: $0.1 - 100\mu\text{m}$** .



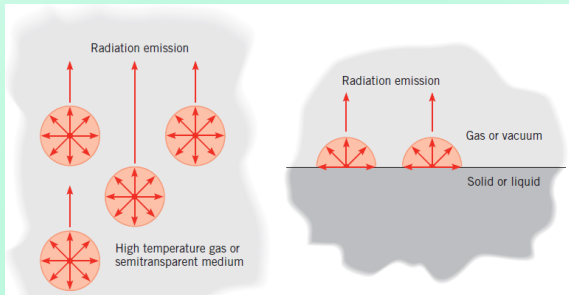
All bodies emit radiation - Radiation at absolute zero temp. is zero

Thermal radiation- Rate at which energy is emitted by matter as a result of its finite temp.



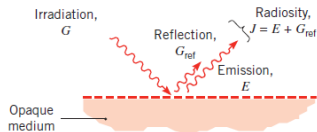
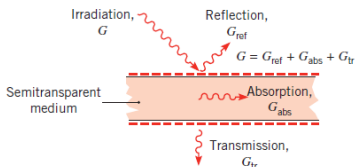
Mechanism of emission - energy released as a result of oscillations of many electrons that constitute matter

- Volumetric phenomenon for semitransparent solids, and gas
 - Surface phenomena in most solids and liquids
- Radiation emitted from interior molecules is strongly absorbed by adjoining molecules
- Radiation emitted from a solid or a liquid originates from molecules within a distance of $1 \mu\text{m}$ from the exposed surface



Over all wavelengths and in all directions

Flux (W/m^2)	Description	Comment
Emissive power, E	Rate at which radiation is emitted from a surface per unit area	$E = \epsilon\sigma T_s^4$
Irradiation, G	Rate at which radiation is incident upon a surface per unit area	Irradiation can be reflected, absorbed, or transmitted
Radiosity, J	Rate at which radiation leaves a surface per unit area	For an opaque surface $J = E + \rho G$
Net radiative flux, $q''_{\text{rad}} = J - G$	Net rate of radiation leaving a surface per unit area	For an opaque surface $q''_{\text{rad}} = \epsilon\sigma T_s^4 - \alpha G$





Reflectivity, ρ is the fraction of the irradiation that is reflected.

Absorptivity, α is the fraction of the irradiation that is absorbed.

Transmissivity, τ is the fraction of the irradiation that is transmitted.

$$\rho + \alpha + \tau = 1$$

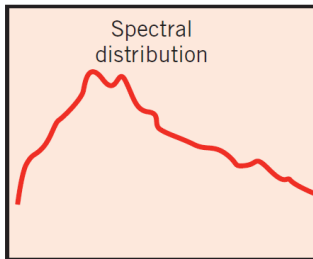
$$J = E + G_{ref} = E + \rho G$$

$$q''_{rad} = J - G = \varepsilon\sigma T_s^4 - \alpha G$$

Spectral distribution and Directionality

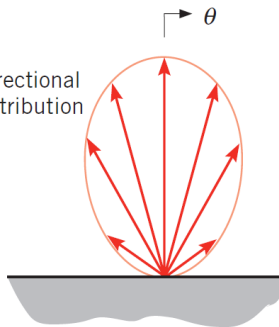
- Emitted radiation is continuous, non-uniform distribution of mono-chromatic (single wavelength) components
- **Spectral distribution** depends on
 - Nature of the emitting surface
 - Temperature of the emitting surface

Monochromatic radiation emission



Wavelength

Directional distribution

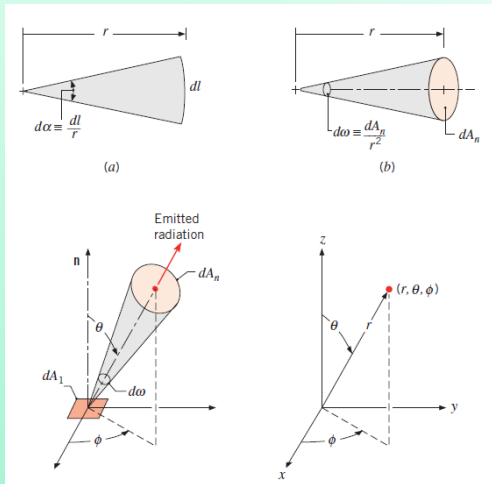




- Radiation emitted by a surface propagates in all directions
- Radiation incident on the surface may come from different directions
- Response of the surface to the radiation depends on the direction
- Directional effects - concept - [Radiation Intensity](#)

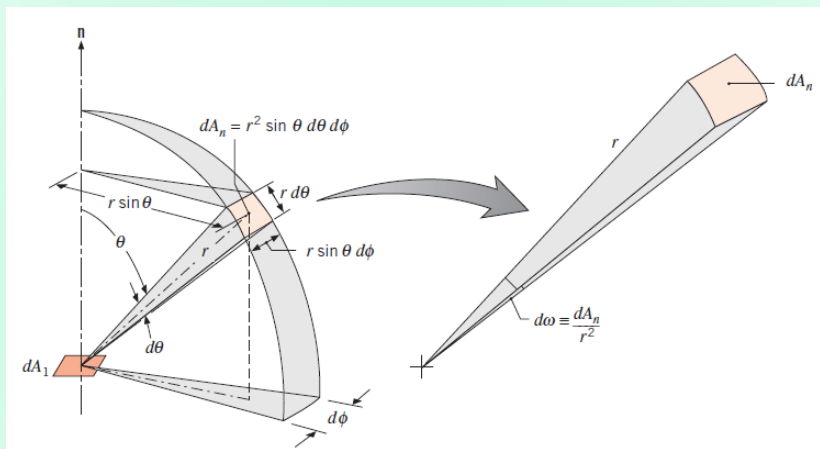
Emission of radiation from a differential area dA_1 into a solid angle $d\omega$ subtended by dA_n at a point on dA_1

$$d\omega \equiv \frac{dA_n}{r^2} \text{ steradian, sr}$$



$$d\omega \equiv \frac{dA_n}{r^2} \quad dA_n = (r d\theta)(r \sin \theta d\phi)$$

$$\therefore d\omega = \sin \theta d\theta d\phi$$





$$\begin{aligned} A_n &= \int_{\text{sphere}} dA_n = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin \theta d\theta d\phi \\ &= 2\pi r^2 \int_{\theta=0}^{\pi} \sin \theta d\theta = 2\pi r^2 \cdot -\cos \theta \Big|_{\theta=0}^{\pi} = 4\pi r^2 \\ A_n &= 4\pi r^2 \end{aligned}$$

For a sphere with unit radius, solid angle is 4π



Differential solid angle $d\omega$ subtended by a differential surface area dA when viewed from a point at a distance r from dA_1 is expressed as

$$d\omega = \frac{dA_n}{r^2} = \frac{dA \cos \alpha}{r^2}$$

where α is the angle between the normal to the surface and the direction of viewing, and thus $dA_n = dA \cos \alpha$ is the normal (or the projected) area to the direction of viewing.



Small surfaces viewed from a relatively large distances can approximately treated as differential areas in solid angle calculations.

For example, the solid angle subtended by 6 cm² plane surface when viewed from a point at a distance of 90 cm along the normal of the surface

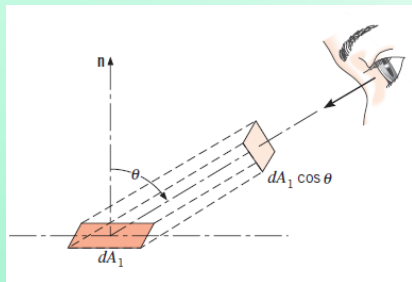
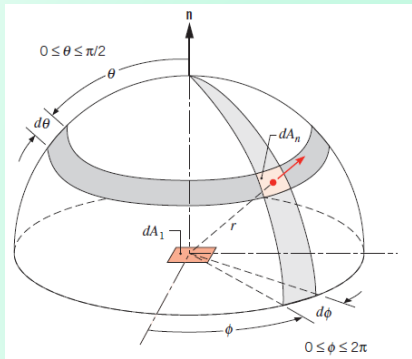
$$\omega = \frac{A_n}{r^2} = \frac{6}{90^2} = 7.41 \times 10^{-4} \text{ sr}$$

If the surface is tilted so that the normal of the surface makes an angle of 60° with the line connecting the point of viewing to the center of the surface, the projected area would be

$$dA_n = dA \cos \alpha = 6 \cos 60^\circ = 3 \text{ cm}^2$$

and the solid angle in this case would be half of the value just determined.

- Radiation streaming through dA_n is $\propto d\omega$ subtended by dA_n
- Radiation is also $\propto dA_1$ as seen by an observer on dA_n , max. at top ($\theta = 0^\circ$) and min. at bottom ($\theta = 90^\circ$)





Radiation intensity for emitted radiation $I_e(\theta, \phi)$ is defined as the rate at which the radiation energy dq is emitted in the (θ, ϕ) direction per unit area normal to this direction and per unit solid angle about this direction.

$$I_e(\theta, \phi) = \frac{dq}{dA_1 \cos \theta \cdot d\omega} = \frac{dq}{dA_1 \cos \theta \cdot \sin \theta d\theta d\phi} \text{ W/m}^2 \text{ sr}$$

Radiation flux for emitted radiation is the **Emissive Power (E)**: rate at which radiation energy is emitted per unit area of the emitting surface which is expressed in the differential form

$$dE = \frac{dq}{dA_1} = I_e(\theta, \phi) \cos \theta \sin \theta d\theta d\phi$$



Emissive power from the surface into hemisphere surrounding it,

$$E = \int_{\text{hemisphere}} dE = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_e(\theta, \phi) \cos \theta \sin \theta d\theta d\phi \text{ W/m}^2$$

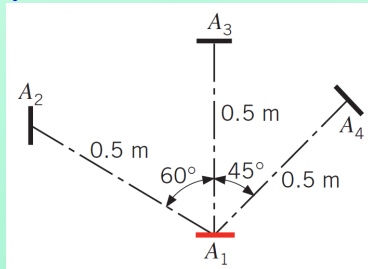
I_e varies with direction (especially with zenith angle θ). Practically approximated as **Diffuse**, i.e., I_e is constant.

$$\begin{aligned} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \cos \theta \sin \theta d\theta d\phi &= 2\pi \int_{\theta=0}^{\pi/2} \cos \theta \sin \theta d\theta \\ &= \frac{2\pi}{2} \int_{\theta=0}^{\pi/2} \sin 2\theta d\theta = \pi \left. \frac{-\cos 2\theta}{2} \right|_0^{\pi/2} = \pi \end{aligned}$$

$$E = \pi I_e$$

Problem

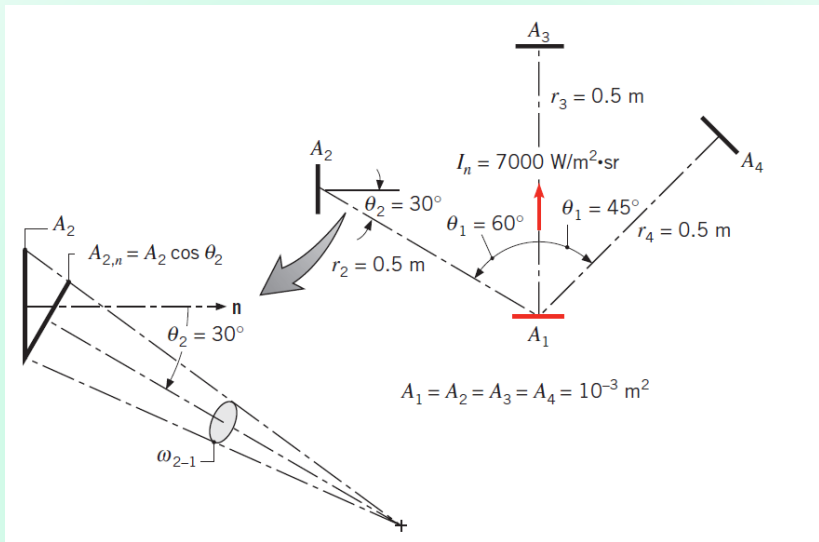
A small surface of area $A_1 = 10^{-3} \text{ m}^2$ is known to emit diffusely and from measurements the total intensity associated with emission in the normal direction is $I_n = 7000 \text{ W/m}^2 \text{ sr}$. Radiation emitted from the surface is intercepted by other surfaces of area $A_2 = A_3 = A_4 = 10^{-3} \text{ m}^2$, which are 0.5 m from A_1 . What is the intensity associated with emission in each of the three directions? What are the solid angles subtended by the three surfaces when viewed from A_1 ? What is the rate at which radiation emitted by A_1 is intercepted by the three surfaces?





Assumptions

- Surface A_1 emits diffusely
- A_1, A_2, A_3, A_4 may be approximated as differential surfaces, $A_j/r_j^2 \ll 1$





Intensity is independent of direction, $I = 7000 \text{ W/m}^2 \text{ sr}$

$$\omega_{3-1} = \omega_{4-1} = \frac{dA_n}{r^2} = \frac{A_3}{r^2} = 4 \times 10^{-3} \text{ sr}$$

$$\omega_{2-1} = \frac{A_2 \cos \theta_2}{r^2} = 3.46 \times 10^{-3} \text{ sr}$$

$$q_{1-j} = I \times A_1 \cos \theta_1 \times \omega_{j-1}$$

$$q_{1-2} = 12.1 \times 10^{-3} \text{ W}$$

$$q_{1-3} = 28.0 \times 10^{-3} \text{ W}$$

$$q_{1-4} = 19.8 \times 10^{-3} \text{ W}$$

Relation to Irradiation

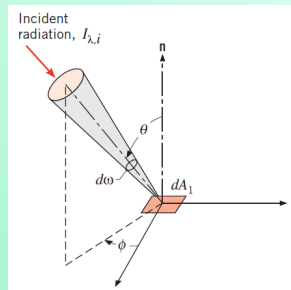
The intensity of incident radiation, $I_i(\theta, \phi)$ is the rate at which radiation energy dG is incident from the (θ, ϕ) direction per unit area of the receiving surface normal to this direction and per unit solid angle about this direction.

$$G_\lambda(\lambda) = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_{\lambda,i}(\lambda, \theta, \phi) \cos \theta \sin \theta d\theta d\phi$$

$$G = \int_0^{\infty} G_\lambda(\lambda) d\lambda$$

If the incident radiation is diffuse,

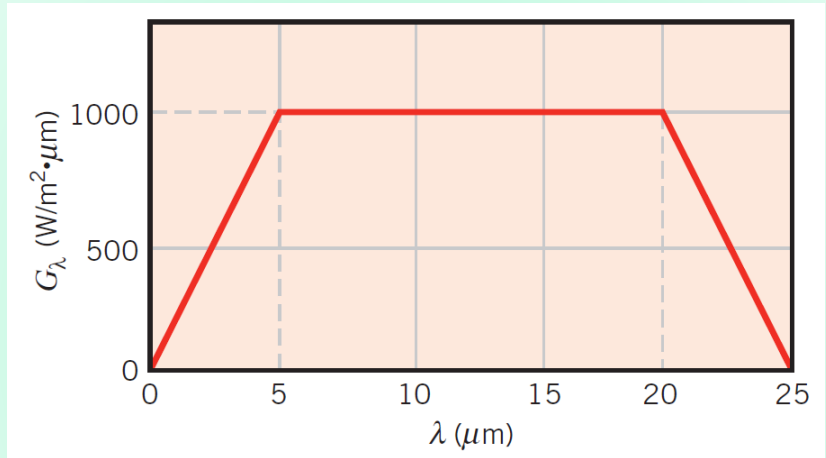
$$G_\lambda = \pi I_{\lambda,i}(\lambda); \quad G = \pi I_i$$



Problem



The spectral distribution of surface irradiation is given below.
What is the total irradiation?



$$G = \int_0^{\infty} G_{\lambda} d\lambda$$

$$G = \int_0^{5 \mu\text{m}} G_{\lambda} d\lambda + \int_{5 \mu\text{m}}^{20 \mu\text{m}} G_{\lambda} d\lambda + \int_{20 \mu\text{m}}^{25 \mu\text{m}} G_{\lambda} d\lambda + \int_{25 \mu\text{m}}^{\infty} G_{\lambda} d\lambda$$



$$J_{\lambda}(\lambda) = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_{\lambda,e+r}(\lambda, \theta, \phi) \cos \theta \sin \theta d\theta d\phi$$

$$J = \int_0^{\infty} J_{\lambda}(\lambda) d\lambda$$

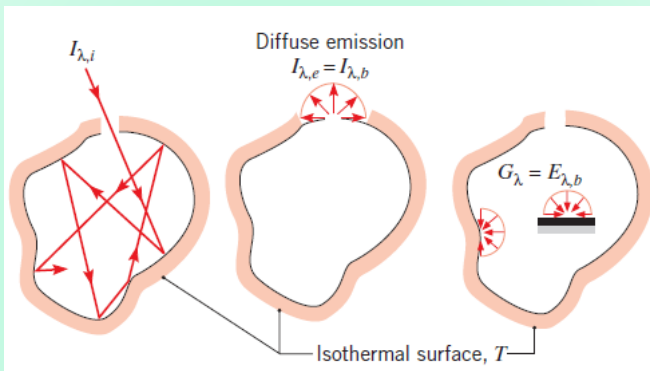
If the surface is diffuse reflector and diffuse emitter,

$$J_{\lambda} = \pi I_{\lambda,e+r}(\lambda); \quad J = \pi I_{e+r}$$

Black Body



- 1 Absorbs all incident radiation, regardless of wavelength and direction.
- 2 For a given T , λ , no surface can emit more energy than a blackbody.
- 3 $I_e \equiv f(\lambda, T)$, independent of direction, **Diffuse emitter**.





The spectral distribution of blackbody emission,

$$I_{\lambda,T} = \frac{2hc_o^2}{\lambda^5 \left[\exp\left(\frac{hc_o}{\lambda kT}\right) - 1 \right]}$$

Planck constant: $h = 6.625 \times 10^{-34}$ J s

Boltzmann constant: $k = 1.3805 \times 10^{-23}$ J/K

The spectral emissive power of a blackbody (diffuse emission)

$$E_{\lambda,T} = \pi I_{\lambda,T} = \frac{C_1}{\lambda^5 \left[\exp\left(\frac{C_2}{\lambda kT}\right) - 1 \right]}$$

The first and second radiation constants are:

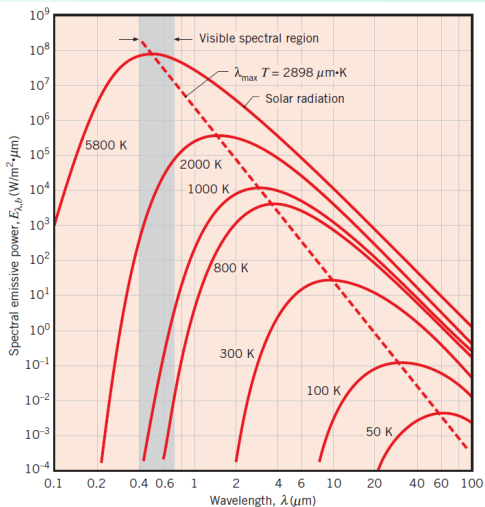
$$C_1 = 2\pi hc_o^2 = 3.742 \times 10^8 \text{ W } \mu\text{m}^4/\text{m}^2$$

$$C_2 = (hc_o/k) = 1.439 \times 10^4 \text{ } \mu\text{m K}$$



- German physicist.
- Studied in Berlin with H.L.F. von Helmholtz and G.R. Kirchhoff, but obtained his doctorate at the University of Munich before returning to Berlin as Prof. in theoretical Physics.
- Later became head of the Kaiser Wilhelm Society (today the Max Planck Institute).
- **Nobel Prize** in Physics in 1918 for his development of the quantum theory.

Spectral Blackbody Emissive Power



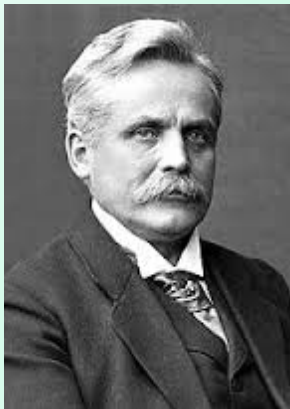
Important Features:

- 1 Emitted radiation varies continuously with λ
- 2 At any λ , E increases with T
- 3 More radiation appearing at shorter λ as T increases
- 4 Significant portion of **solar radiation (5800 K)** is in visible region
- 5 For $T \lesssim 800$ K, emission is predominantly in the infrared region of the spectrum and is not visible to the eye.



$$\lambda_{\max}T = C_3 = 2897.8 \mu\text{m K}$$

- 1 Maximum spectral power is displaced to shorter λ with increasing T
- 2 Solar radiation - middle of the spectrum ($\lambda = 0.5 \mu\text{m}$), since 5800 K
- 3 Blackbody at 1000 K, peak emission - $2.9 \mu\text{m}$
- 4 With increasing T , shorter λ become more prominent, until eventually significant emission occurs over the entire visible spectrum
- 5 Tungsten filament lamp - 2900 K ($\lambda_{\max} = 1.0 \mu\text{m}$) emits white light, although most of the emission remains in infrared region



- German physicist.
- Prof. of Physics at the University of Giessen and later at the University of Munich.
- Besides his research in the area of electromagnetic waves, his interests included other rays, such as electron beams, X-rays, and α -particles.
- **Nobel Prize** in Physics in 1911 for his discovery of his displacement law.



$$E_b = \int_0^{\infty} \frac{C_1}{\lambda^5 \left[\exp\left(\frac{C_2}{\lambda k T}\right) - 1 \right]} d\lambda = \sigma T^4 \text{ W/m}^2$$

$$\sigma = 5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4$$

This enables calculation of the amount of radiation emitted in all directions and over all wavelengths simply from the knowledge of the temperature of the blackbody.

The total intensity associated with blackbody emission is:

$$I_b = \frac{E_b}{\pi} \text{ W/m}^2 \text{ sr}$$



Joseph Stefan (1835-1893)

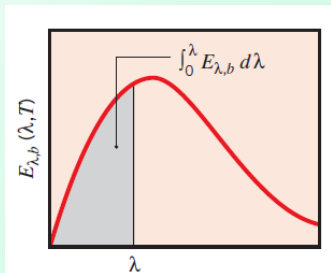
- Gymnastics, Poet, **Austrian Physicist**.
- Prof. at the University of Vienna
- $E_b \propto T^4$ from experiments in 1879

Ludwig Erhard Boltzmann (1844-1906)

- Austrian Physicist. Doctorate from the University of Vienna. Prof. in Vienna, Graz, Munich and Leipzig (Germany).
- His greatest contributions were in the field of statistical mechanics (Boltzmann statistics).
- Derived from thermodynamics in 1889.



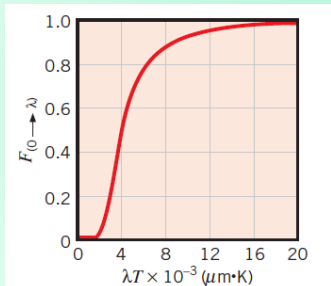
John Crepeau, Josef Stefan: His Life and Legacy in the Thermal Sciences, *Experimental Thermal and Fluid Science*, 31 (2007) 795-803.



$$F_{(0-\lambda)} \equiv \frac{\int_0^\lambda E_{\lambda,b} d\lambda}{\int_0^\infty E_{\lambda,b} d\lambda} = \frac{\int_0^\lambda E_{\lambda,b} d\lambda}{\sigma T^4}$$

$$\lambda T = m; d\lambda T = dm; d\lambda = dm/T$$

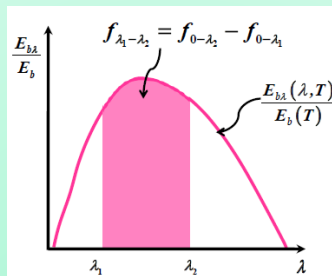
$$\lambda = 0; m = 0$$



$$F_{(0-\lambda)} \equiv \int_0^{\lambda T} \frac{E_{\lambda,b} d(\lambda T)}{\sigma T^4 T} = \int_0^{\lambda T} \frac{E_{\lambda,b} d(\lambda T)}{\sigma T^5}$$

$$= f(\lambda T)$$

$$F_{(\lambda_1-\lambda_2)} = \frac{\int_0^{\lambda_2} E_{\lambda,b} d\lambda - \int_0^{\lambda_1} E_{\lambda,b} d\lambda}{\sigma T^4} = F_{(0-\lambda_2)} - F_{(0-\lambda_1)}$$



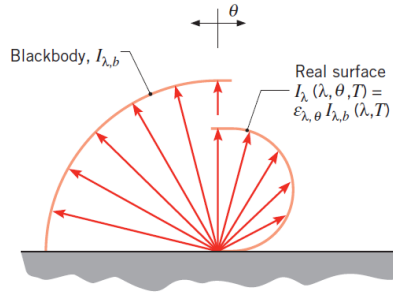
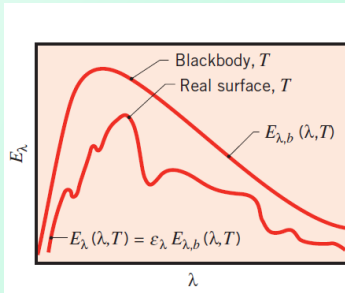
Since the integrand ($E_{\lambda,b}/\sigma T^4$) is exclusively a function of wavelength temperature product (λT). The results are presented in tabular form to obtain $F_{0-\lambda}$ as a function of λT .

Blackbody Radiation Functions



λT ($\mu\text{m} \cdot \text{K}$)	$F_{(0 \rightarrow \lambda)}$	$I_{\lambda,b}(\lambda, T)/\sigma T^5$ ($\mu\text{m} \cdot \text{K} \cdot \text{sr}$) ⁻¹	$\frac{I_{\lambda,b}(\lambda, T)}{I_{\lambda,b}(\lambda_{\text{max}}, T)}$
200	0.000000	0.375034×10^{-27}	0.000000
400	0.000000	0.490335×10^{-13}	0.000000
600	0.000000	0.104046×10^{-8}	0.000014
800	0.000016	0.991126×10^{-7}	0.001372
1,000	0.000321	0.118505×10^{-5}	0.016406
1,200	0.002134	0.523927×10^{-5}	0.072534
1,400	0.007790	0.134411×10^{-4}	0.186082
1,600	0.019718	0.249130	0.344904
1,800	0.039341	0.375568	0.519949
2,000	0.066728	0.493432	0.683123
2,200	0.100888	0.589649×10^{-4}	0.816329
2,400	0.140256	0.658866	0.912155
2,600	0.183120	0.701292	0.970891
2,800	0.227897	0.720239	0.997123
2,898	0.250108	0.722318×10^{-4}	1.000000

Emission from Real Surfaces





$$\varepsilon(T) = \frac{E(T)}{E_b(T)} \implies E(T) = \varepsilon(T)\sigma T^4$$

Spectral, directional emissivity,

$$\varepsilon_{\lambda,\theta}(\lambda, \theta, \phi, T) = \frac{I_{\lambda,e}(\lambda, \theta, \phi, T)}{I_{\lambda,b}(\lambda, T)}$$

Total, directional emissivity,

$$\varepsilon_{\theta}(\theta, \phi, T) = \frac{I_e(\theta, \phi, T)}{I_b(T)}$$

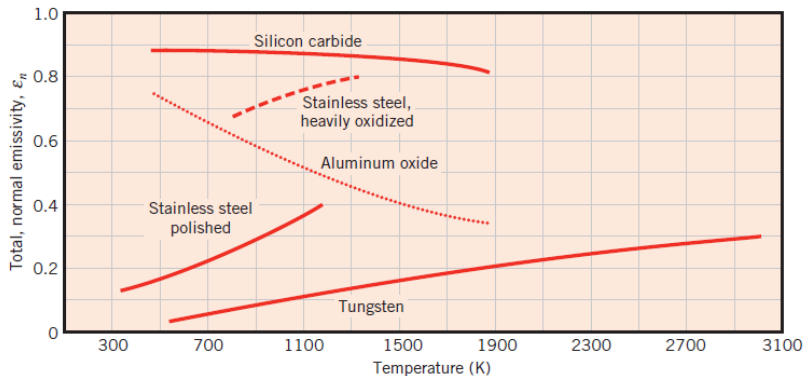
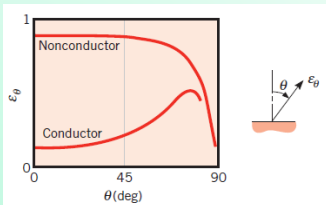
Spectral, hemispherical emissivity,

$$\varepsilon_{\lambda}(\lambda, T) = \frac{E_{\lambda}(\lambda, T)}{E_{\lambda,b}(\lambda, T)}$$

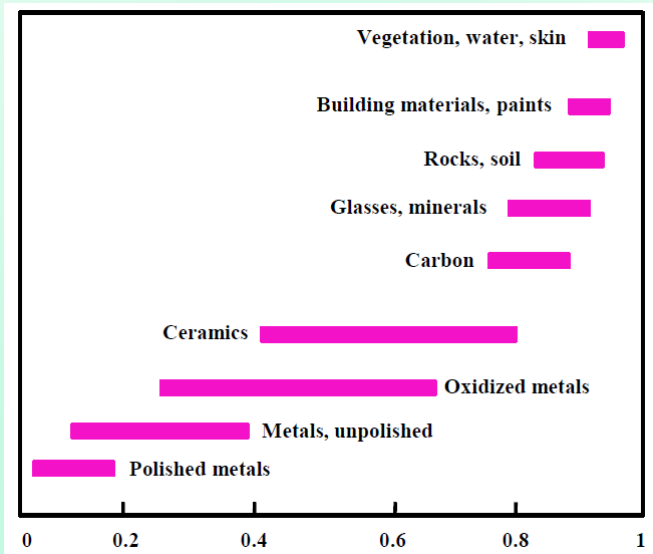
Emission from Real Surfaces



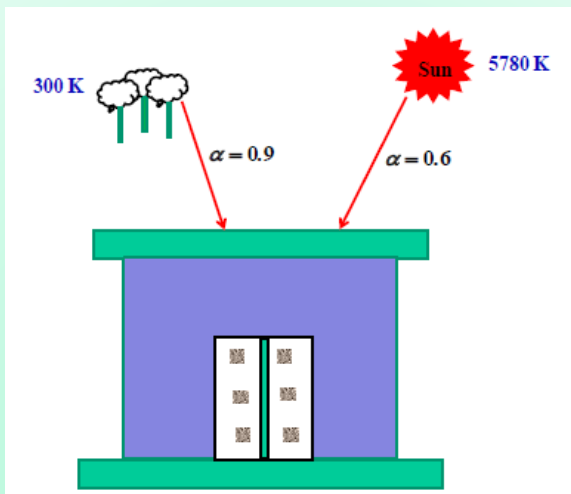
$$\varepsilon(T) = \frac{\int_0^{\infty} \varepsilon_{\lambda}(\lambda, T) E_{\lambda, b}(\lambda, T)}{E_b(T)}$$



Emission from Real Surfaces



Absorptivity



α depends strongly on T_{source} at which the incident radiation is originating.

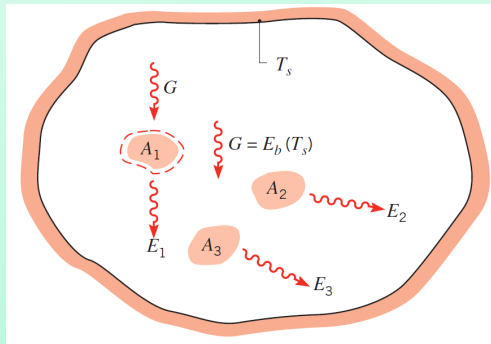


- $T_{sur} < 1000$ K - Emission is in IR region - not seen by eye.
- Color is due to selective reflection and absorption of the visible portion of the irradiation.
- Leaf is green - Chlorophyll - absorbs the blue, red and preferential reflection in the green.
- Color of the Surface - not an indicator of the overall capacity of an absorber or reflector, since much of the irradiation may be in the IR region.
- Surface coated with lamp black/soot appears to be black to the human eye. Also thermally black. Snow is 'white' to the eye but **black** for the thermal radiation.

Kirchhoff's Law



Consider a large, isothermal enclosure of T_s within which several small bodies are confined.



Regardless of its orientation, the irradiation experienced by any body in the cavity is diffuse and equal to emission from a blackbody at T_s .



Under steady state conditions, thermal equilibrium must exist, $T_1 = T_2 = T_3 = \dots = T_s$ and the net rate of energy transfer to each surface must be zero.

Applying energy balance to a control surface about body 1,

$$\alpha_1 G A_1 - E_1(T_s) A_1 = 0$$

$$\alpha_1 E_b(T_s) - E_1(T_s) = 0 \quad \because G = E_b(T_s)$$
$$\frac{E_1(T_s)}{\alpha_1} = \frac{E_2(T_s)}{\alpha_2} = \frac{E_3(T_s)}{\alpha_3} = \dots = E_b(T_s)$$

This relation is known as [Kirchhoff's law](#). No real surface can have an emissive power exceeding that of a black surface at the same temperature, and the notion of the black body as an ideal emitter is confirmed.



$$\frac{\varepsilon_1}{\alpha_1} = \frac{\varepsilon_2}{\alpha_2} = \frac{\varepsilon_3}{\alpha_3} = \dots = 1$$

Hence, for any surface in the enclosure, $\varepsilon = \alpha$.

Total hemispherical emissivity = total hemispherical absorptivity.

The restrictive conditions inherent in this derivation is:

- the surface irradiation has been assumed to correspond to emission from a blackbody at the same temperature as the surface.

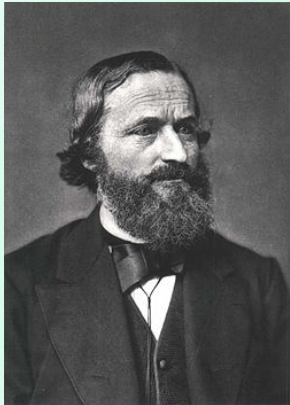


$\varepsilon_\lambda(T) = \alpha_\lambda(T)$, valid when G is independent of direction.

The form of the Kirchhoff's law that involves no restrictions is the **spectral directional** form,

$$\varepsilon_{\lambda,\theta}(T) = \alpha_{\lambda,\theta}(T)$$

- It is very tempting to use Kirchhoff's law in radiation analysis
∴ the relation $\varepsilon = \alpha$ together with $\rho = 1 - \alpha$ enables us to determine all three properties of a opaque surface from a knowledge of only one property.
- Although, $\varepsilon = \alpha$ gives acceptable results in most cases, in practice, **care** should be exercised when there is **considerable difference between T_{surface} and T_{source}** .



- German physicist.
- Formulated his circuit laws, ubiquitous in electrical engineering in 1845, while still a student.
- He completed this study as a seminar exercise, it later became his doctoral dissertation.
- Proposed his law of thermal radiation in 1859, and gave proof in 1861.
- He was called to the University of Heidelberg in 1854, where he collaborated in Spectroscopic work with **Robert Bunsen**. Together, they discovered Caesium ($z = 37$) and Rubidium ($z = 55$) in 1861.



- Properties are independent of λ .
- **Diffuse surface** is a surface in which its properties are independent of direction.
- Emissivity of a gray, diffuse surface is total hemispherical emissivity of that surface because of the independence of λ and direction.



The irradiance at the earth's surface due to atmospheric emission,

$$G_{atm} = \sigma T_{sky}^4$$

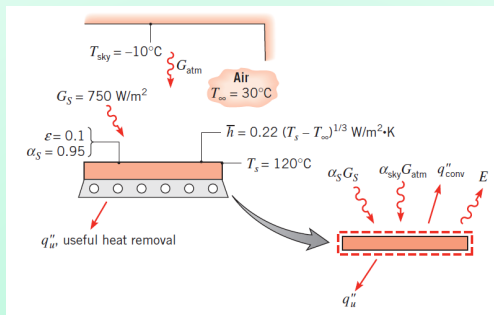
T_{sky} is the **effective sky temperature**
= 275 K, moderate temp.
= 230 K, cold, clear sky
= 285 K, warm, cloudy sky



A flat-plate solar collector with no cover plate has a selective absorber surface of emissivity 0.1 and solar absorptivity 0.95. At a given time of day the absorber surface temperature T_s is 120°C when the solar irradiation is 750 W/m^2 , the effective sky temperature is -10°C , and the ambient air temperature T_∞ is 30°C . Assume that the heat transfer coefficient for the calm day conditions can be estimated from,

$$\bar{h} = 0.22(T_s - T_\infty)^{1/3} \text{ W/m}^2 \text{ K}$$

Calculate the useful heat removal rate (W/m^2) from the collector for these conditions. What is the corresponding efficiency of the collector?

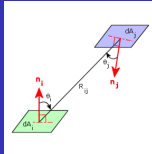


$$\alpha_s G_s + \alpha_{\text{sky}} G_{\text{atm}} - q''_{\text{conv}} - E - q''_u = 0$$

$$\alpha_s G_s + \varepsilon \sigma T_{\text{sky}}^4 - \bar{h}(T_s - T_\infty) - \varepsilon \sigma T_s^4 - q''_u = 0$$

$$q''_u = 516 \text{ W/m}^2; \quad \eta = \frac{q''_u}{G_s} = 0.69$$

Heat and Mass Transfer



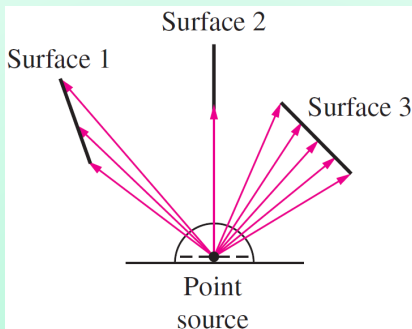
View Factor

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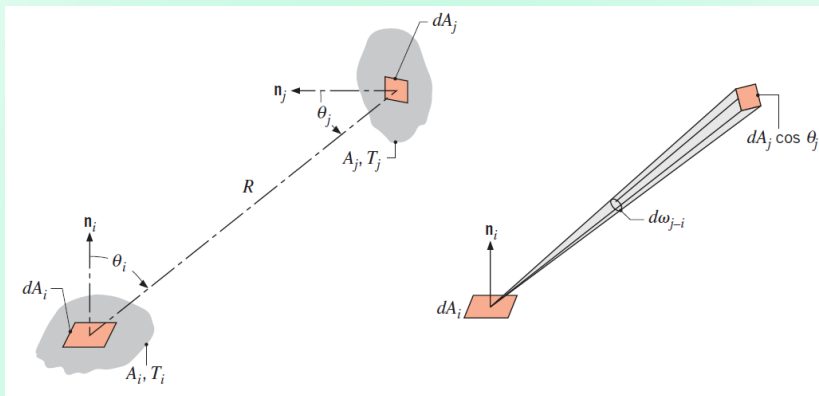


Department of Mechanical Engineering
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- Radiation heat transfer between surfaces depends on the orientation of the surfaces relative to each other as well as their radiation properties and temperatures.
- This dependence is accounted for by the **view factor**.
- By facing the fire from front or back - maximum radiation
- By facing the fire from the side - minimum radiation

F_{ij} - the fraction of the radiation leaving surface i that strikes the surface j directly.





The rate at which radiation leaves dA_i and is intercepted by dA_j is,

$$dq_{i \rightarrow j} = I_{e+r,i} \cos \theta_i dA_i d\omega_{j-i}$$

$I_{e+r,i}$ - intensity of radiation leaving surface i , and $d\omega_{j-i}$ - solid angle subtended by dA_j when viewed from dA_i : $d\omega_{j-i} = \frac{\cos \theta_j dA_j}{R^2}$

$$dq_{i \rightarrow j} = I_{e+r,i} \frac{\cos \theta_i \cos \theta_j}{R^2} dA_i dA_j$$

Assuming that surface i emits and reflects diffusely,

$$dq_{i \rightarrow j} = J_i \frac{\cos \theta_i \cos \theta_j}{\pi R^2} dA_i dA_j$$

The total rate of radiation leaving surface i towards j ,

$$q_{i \rightarrow j} = J_i \int_{A_i} \int_{A_j} \frac{\cos \theta_i \cos \theta_j}{\pi R^2} dA_i dA_j$$



$$F_{ij} = \frac{q_{i \rightarrow j}}{A_i J_i}$$

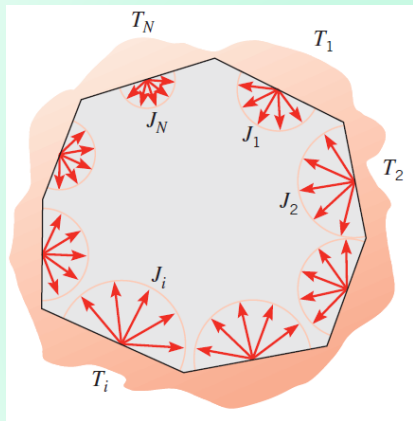
$$F_{ij} = \frac{1}{A_i} \int_{A_i} \int_{A_j} \frac{\cos \theta_i \cos \theta_j dA_j}{\pi R^2} dA_i dA_j$$

$$F_{ji} = \frac{1}{A_j} \int_{A_i} \int_{A_j} \frac{\cos \theta_i \cos \theta_j dA_j}{\pi R^2} dA_i dA_j$$

These equations may be used to determine the view factor associated with any two surfaces that are **diffuse emitters** and **reflectors** and have **uniform radiosity**.

$$A_i F_{ij} = A_j F_{ji}$$

Reciprocity relation



For an enclosure,

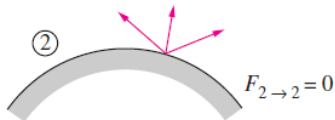
$$\sum_{j=1}^N F_{ij} = 1$$

Summation rule

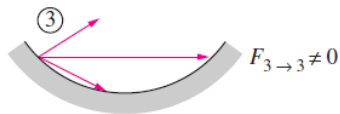
From the conservation requirement that all radiation leaving surface i must be intercepted by the enclosure surfaces.



(a) Plane surface



(b) Convex surface



(c) Concave surface

To calculate radiation exchange in an enclosure of N surfaces, a total of N^2 view factors is needed.

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} & \dots & F_{1N} \\ F_{21} & F_{22} & F_{23} & \dots & F_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{N1} & F_{N2} & F_{N3} & \dots & F_{NN} \end{bmatrix}$$

N eqs. from summation rule.

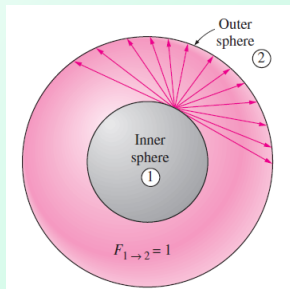
$N(N - 1)/2$ eqs. from reciprocity.

A total of

$$N^2 - (N + N(N - 1)/2) =$$

$N(N - 1)/2$ view factors need to be determined directly.

View Factor Relations - Example



$n^2 = 4$ view factors ($F_{11}, F_{12}, F_{21}, F_{22}$).
 $N(N - 1)/2 = 1$ to be determined directly.

$$F_{12} = 1$$

$$F_{21} = \frac{A_1}{A_2} F_{12} = \frac{A_1}{A_2}$$

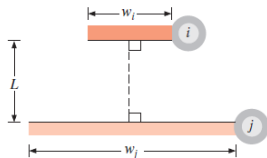
$$F_{11} + F_{12} = 1 \quad \implies \quad F_{11} = 0$$

$$F_{21} + F_{22} = 1 \quad F_{22} = 1 - \frac{A_1}{A_2}$$

Geometry

Relation

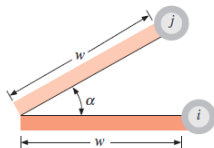
Parallel Plates with Midlines Connected by Perpendicular



$$F_{ij} = \frac{[(W_i + W_j)^2 + 4]^{1/2} - [(W_j - W_i)^2 + 4]^{1/2}}{2W_i}$$

$$W_i = w_i/L, W_j = w_j/L$$

Inclined Parallel Plates of Equal Width and a Common Edge

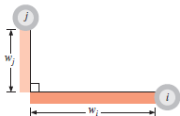


$$F_{ij} = 1 - \sin\left(\frac{\alpha}{2}\right)$$

Geometry

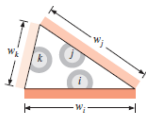
Relation

Perpendicular Plates with a Common Edge



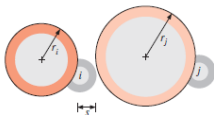
$$F_{ij} = \frac{1 + (w_j/w_i) - [1 + (w_j/w_i)^2]^{1/2}}{2}$$

Three-Sided Enclosure



$$F_{ij} = \frac{w_i + w_j - w_k}{2w_i}$$

Parallel Cylinders of Different Radii

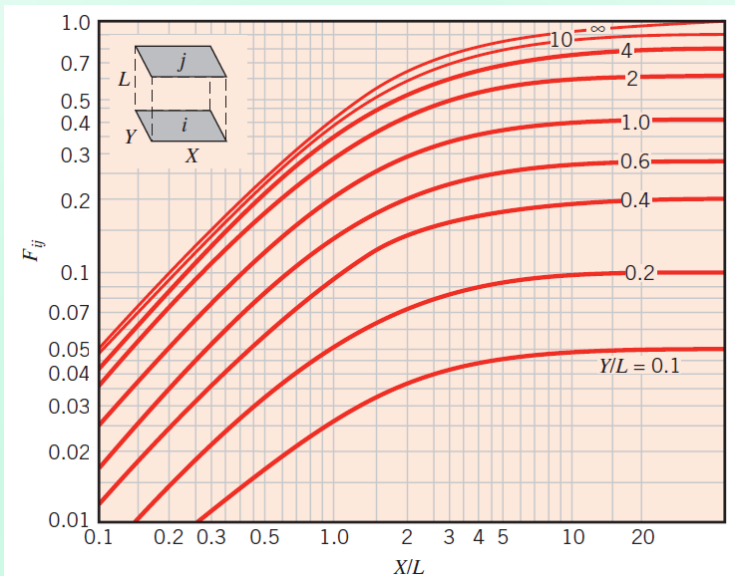


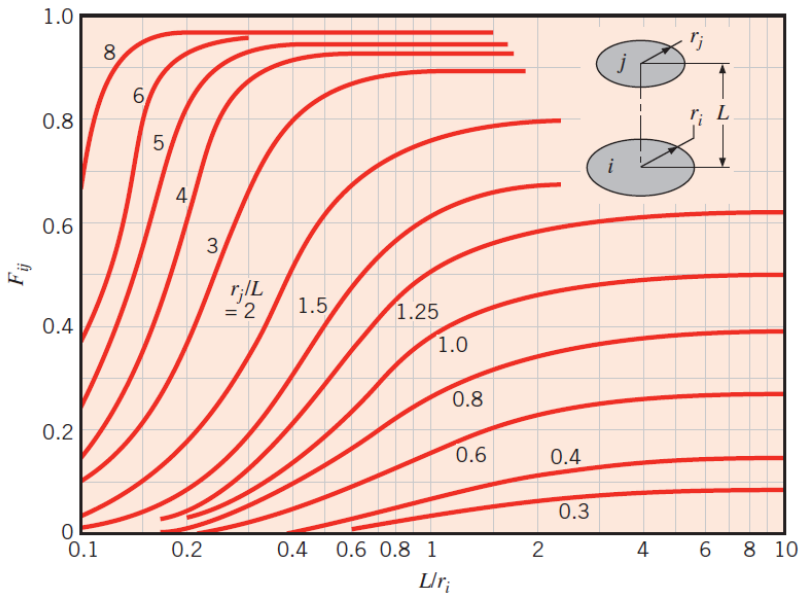
$$F_{ij} = \frac{1}{2\pi} \left\{ \pi + [C^2 - (R + 1)^2]^{1/2} - [C^2 - (R - 1)^2]^{1/2} + (R - 1) \cos^{-1} \left[\left(\frac{R}{C} \right) - \left(\frac{1}{C} \right) \right] - (R + 1) \cos^{-1} \left[\left(\frac{R}{C} \right) + \left(\frac{1}{C} \right) \right] \right\}$$

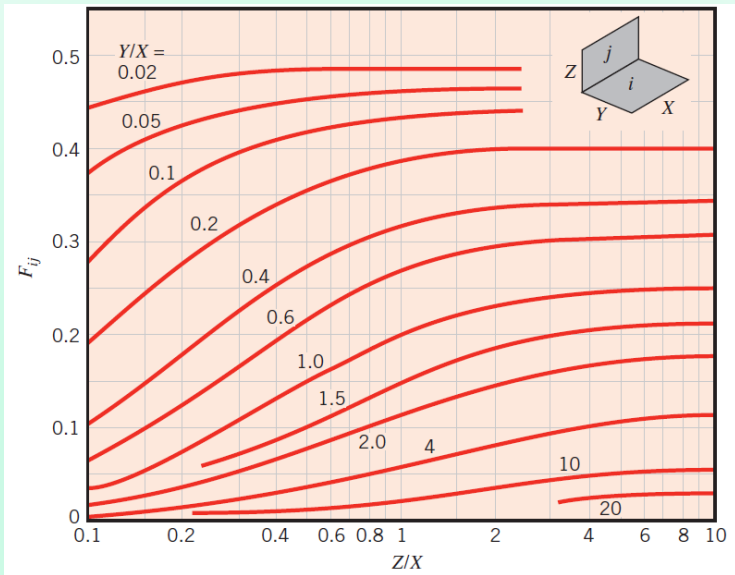
$$R = r_j/r_i, S = s/r_i$$

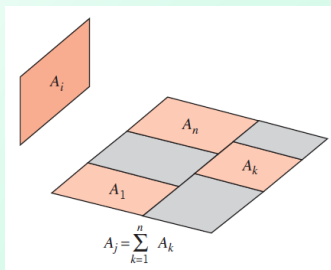
$$C = 1 + R + S$$

Geometry	Relation
<p>Aligned Parallel Rectangles</p>	$\bar{X} = X/L, \bar{Y} = Y/L$ $F_{ij} = \frac{2}{\pi \bar{X} \bar{Y}} \left\{ \ln \left[\frac{(1 + \bar{X}^2)(1 + \bar{Y}^2)}{1 + \bar{X}^2 + \bar{Y}^2} \right]^{1/2} \right.$ $+ \bar{X}(1 + \bar{Y}^2)^{1/2} \tan^{-1} \frac{\bar{X}}{(1 + \bar{Y}^2)^{1/2}}$ $\left. + \bar{Y}(1 + \bar{X}^2)^{1/2} \tan^{-1} \frac{\bar{Y}}{(1 + \bar{X}^2)^{1/2}} - \bar{X} \tan^{-1} \bar{X} - \bar{Y} \tan^{-1} \bar{Y} \right\}$
<p>Coaxial Parallel Disks</p>	$R_i = r_i/L, R_j = r_j/L$ $S = 1 + \frac{1 + R_j^2}{R_i^2}$ $F_{ij} = \frac{1}{2} \{ S - [S^2 - 4(R_j/r_i)^2]^{1/2} \}$
<p>Perpendicular Rectangles with a Common Edge</p>	$H = Z/X, W = Y/X$ $F_{ij} = \frac{1}{\pi W} \left(W \tan^{-1} \frac{1}{W} + H \tan^{-1} \frac{1}{H} \right.$ $- (H^2 + W^2)^{1/2} \tan^{-1} \frac{1}{(H^2 + W^2)^{1/2}}$ $+ \frac{1}{4} \ln \left\{ \frac{(1 + W^2)(1 + H^2)}{1 + W^2 + H^2} \left[\frac{W^2(1 + W^2 + H^2)}{(1 + W^2)(W^2 + H^2)} \right]^{W^2} \right.$ $\left. \times \left[\frac{H^2(1 + H^2 + W^2)}{(1 + H^2)(H^2 + W^2)} \right]^{H^2} \right\}$









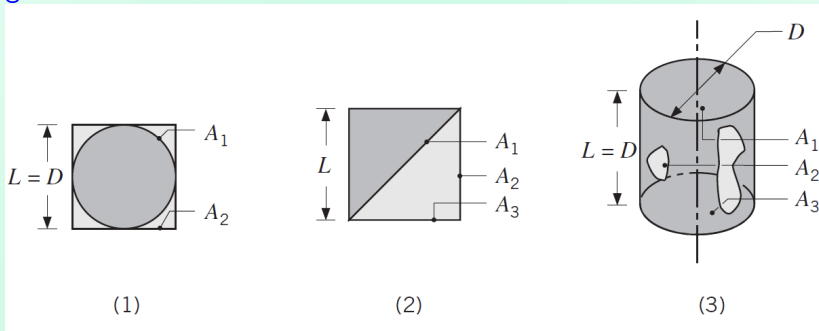
$$F_{i(j)} = \sum_{k=1}^n F_{i,k}$$

$$A_j F_{(j)i} = \sum_{k=1}^n A_k F_{k,i}$$

$$F_{(j)i} = \frac{\sum_{k=1}^n A_k F_{k,i}}{\sum_{k=1}^n A_k}$$

This may be applied when the originating surface is composed of several parts.

Determine the view factors F_{12} and F_{21} for the following geometries:



- 1 Sphere of diameter D inside a cubical box of length $L = D$.
- 2 One side of a diagonal partition within a long square duct.
- 3 End and side of a circular tube of equal length and diameter.

Assumptions: Diffuse surfaces with uniform radiosities.

- ① Sphere within a cube: $F_{12} = 1$

$$F_{21} = \frac{A_1}{A_2} F_{12} = \frac{\pi}{6}$$

- ② Partition within a square duct: ~~F_{11}~~ ⁰ + $F_{12} + F_{13} = 1$

$$F_{12} = F_{13} = 0.5 \quad \text{from symmetry}$$

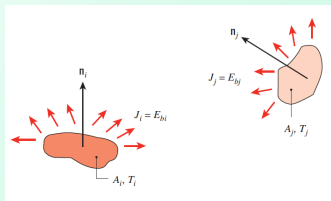
$$F_{21} = \frac{A_1}{A_2} F_{12} = 0.71$$

- ③ Circular tube: $F_{ij}(L/r_i, r_j/L) = F_{ij}(2, 0.5) \approx 0.17$

$$\del{F_{11}}^{\color{red}0} + F_{12} + F_{13} = 1 \implies F_{12} = 0.83$$

$$F_{21} = \frac{A_1}{A_2} F_{12} = 0.21$$

Blackbody Radiation Exchange



$$q_{i \rightarrow j} = (A_i J_i) F_{ij}$$

For a blackbody, $J_i = E_{bi}$,

$$q_{i \rightarrow j} = A_i F_{ij} E_{bi}$$

Similarly, $q_{j \rightarrow i} = A_j F_{ji} E_{bj}$

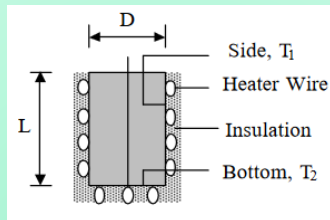
The **net radiative exchange** between the two surfaces,

$$q_{ij} = q_{i \rightarrow j} - q_{j \rightarrow i} = A_i F_{ij} \sigma (T_i^4 - T_j^4)$$

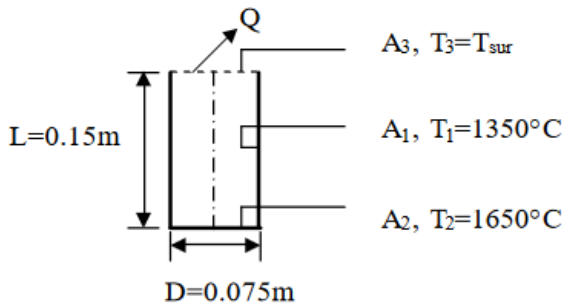
q_{ij} is the net rate at which radiation leaves i as a result of its interaction with j = the net rate at which j gains radiation due to its interaction with i .

For an enclosure of black surfaces, $q_i = \sum_{j=1}^N A_i F_{ij} \sigma (T_i^4 - T_j^4)$

A furnace cavity, which is in the form of a cylinder of 75 mm diameter and 150 mm length, is open at one end to large surroundings that are at 27°C . The sides and bottom may be approximated as blackbodies, are heated electrically, are well insulated, and are maintained at temperatures of 1350 and 1650°C , respectively. How much power is required to maintain the furnace conditions?



$$T_{\text{sur}} = 300\text{K}$$





$$\begin{aligned}q &= q_{13} + q_{23} \\ &= A_1 F_{13} \sigma (T_1^4 - T_3^4) + A_2 F_{23} \sigma (T_2^4 - T_3^4)\end{aligned}$$

$$F_{ij}(L/r_i, r_j/L) = F_{ij}(4, 0.25) \approx 0.06$$

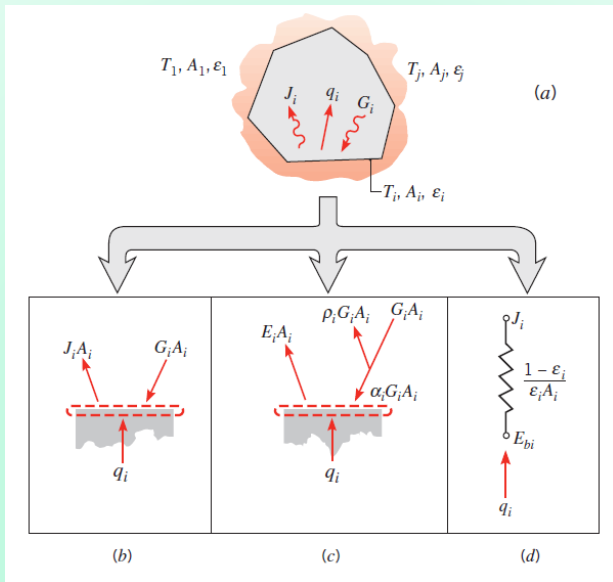
From summation rule, $F_{21} = 1 - F_{23} = 0.94$

From reciprocity, $F_{12} = \frac{A_2}{A_1} F_{21} = 0.118$

From symmetry, $F_{13} = F_{12}$

$$q = 1639 + 205 = 1844 \text{ W}$$

Diffuse, Gray Surfaces in an Enclosure





$$q_i = A_i(J_i - G_i)$$

$$J_i \equiv E_i + \rho_i G_i$$

For an opaque, diffuse, gray surface,

$$J_i = \varepsilon_i E_{bi} + (1 - \varepsilon_i)G_i$$

$$q_i = \frac{E_{bi} - J_i}{(1 - \varepsilon_i)/\varepsilon_i A_i}$$

$(E_{bi} - J_i)$ is the driving potential with a **surface radiative resistance** of the form $(1 - \varepsilon_i)/\varepsilon_i A_i$.

$$J_i = ?$$



$$A_i G_i = \sum_{j=1}^N F_{ji} A_j J_j = \sum_{j=1}^N F_{ij} A_i J_j$$

We have, $q_i = A_i(J_i - G_i)$

$$\Rightarrow q_i = A_i \left(J_i - \sum_{j=1}^N F_{ij} J_j \right)$$

From summation rule,

$$q_i = A_i \left(\sum_{j=1}^N F_{ij} J_i - \sum_{j=1}^N F_{ij} J_j \right)$$

$$q_i = \sum_{j=1}^N A_i F_{ij} (J_i - J_j) = \sum_{j=1}^N q_{ij}$$



If T_i is known,

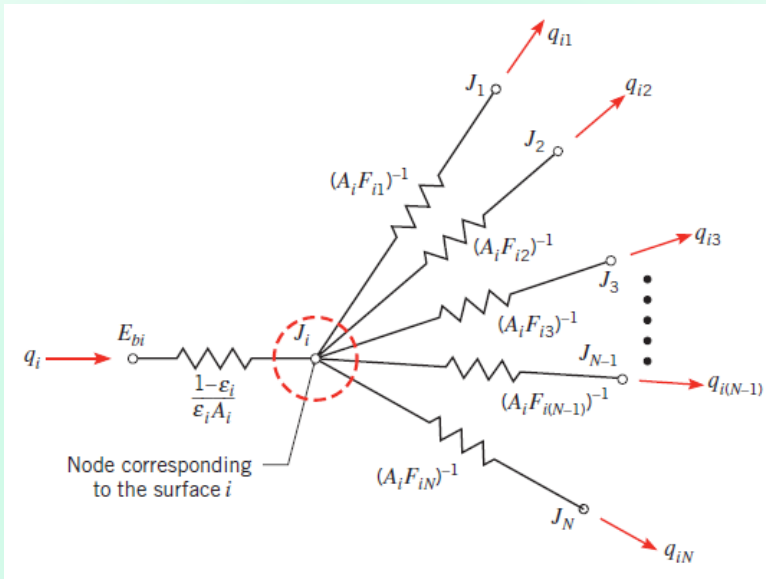
$$\frac{E_{bi} - J_i}{(1 - \varepsilon_i)/\varepsilon_i A_i} = \sum_{j=1}^N \frac{(J_i - J_j)}{(A_i F_{ij})^{-1}}$$

If q_i is known,

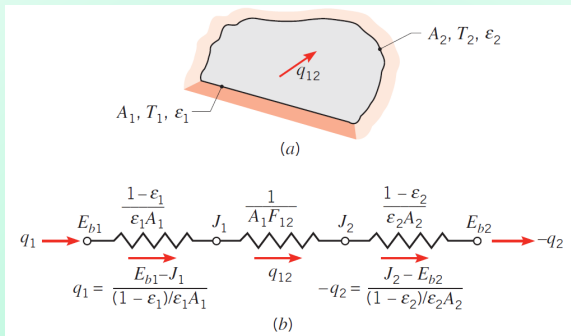
$$q_i = \sum_{j=1}^N \frac{(J_i - J_j)}{(A_i F_{ij})^{-1}}$$

Solve N equations for N unknowns, $J_1, J_2, J_3, \dots, J_N$.

Radiative Exchange Between Surfaces



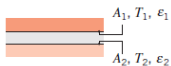
Two-Surface Enclosure



$$q_{12} = q_1 = -q_2 = \frac{\sigma(T_1^4 - T_2^4)}{\frac{1 - \varepsilon_1}{\varepsilon_1 A_1} + \frac{1}{A_1 F_{12}} + \frac{1 - \varepsilon_2}{\varepsilon_2 A_2}}$$

This can be used for any two diffuse, gray surfaces that form an enclosure.

Large (Infinite) Parallel Planes

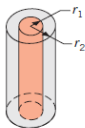


$$A_1 = A_2 = A$$

$$F_{12} = 1$$

$$q_{12} = \frac{A\sigma(T_1^4 - T_2^4)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1}$$

Long (Infinite) Concentric Cylinders

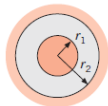


$$\frac{A_1}{A_2} = \frac{r_1}{r_2}$$

$$F_{12} = 1$$

$$q_{12} = \frac{\sigma A_1(T_1^4 - T_2^4)}{\frac{1}{\epsilon_1} + \frac{1 - \epsilon_2}{\epsilon_2} \left(\frac{r_1}{r_2}\right)}$$

Concentric Spheres

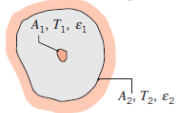


$$\frac{A_1}{A_2} = \frac{r_1^2}{r_2^2}$$

$$F_{12} = 1$$

$$q_{12} = \frac{\sigma A_1(T_1^4 - T_2^4)}{\frac{1}{\epsilon_1} + \frac{1 - \epsilon_2}{\epsilon_2} \left(\frac{r_1}{r_2}\right)^2}$$

Small Convex Object in a Large Cavity

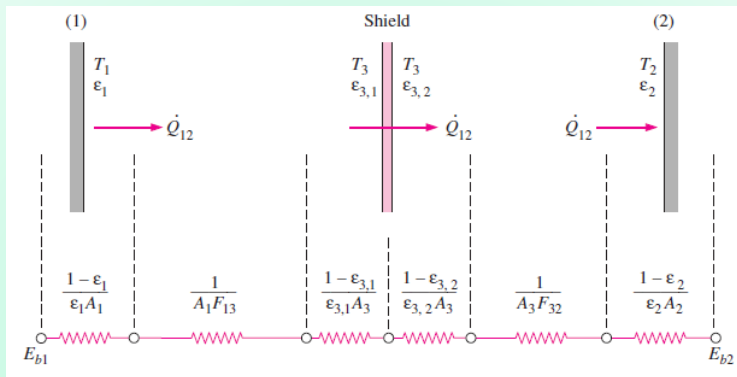


$$\frac{A_1}{A_2} \approx 0$$

$$F_{12} = 1$$

$$q_{12} = \sigma A_1 \epsilon_1 (T_1^4 - T_2^4)$$

Radiation Shields



$$q_{12, \text{no shield}} = \frac{A_1 \sigma (T_1^4 - T_2^4)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1}$$

$$q_{12} = \frac{E_{b,1} - E_{b,2}}{\frac{1-\epsilon_1}{\epsilon_1} + \frac{1}{A_1 F_{1,2}} + \frac{1-\epsilon_{3,1}}{\epsilon_{3,1}} + \frac{1-\epsilon_{3,2}}{\epsilon_{3,2}} + \frac{1}{A_3 F_{3,2}} + \frac{1-\epsilon_2}{\epsilon_2}}$$



$$F_{13} = F_{32} = 1$$

$$q_{12} = \frac{A_1 \sigma (T_1^4 - T_2^4)}{\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1\right) + \left(\frac{1}{\varepsilon_{3,1}} + \frac{1}{\varepsilon_{3,2}} - 1\right)}$$

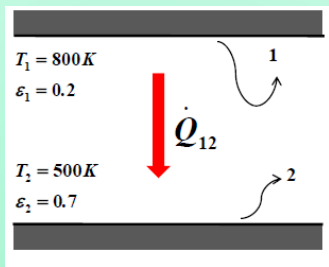
$$(q_{12})_N = \frac{A_1 \sigma (T_1^4 - T_2^4)}{\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1\right) + \left(\frac{1}{\varepsilon_{3,1}} + \frac{1}{\varepsilon_{3,2}} - 1\right) + \dots + \left(\frac{1}{\varepsilon_{N,1}} + \frac{1}{\varepsilon_{N,2}} - 1\right)}$$

If all emissivities are equal,

$$(q_{12})_N = \frac{1}{N+1} (q_{12})_0$$

$(q_{12})_0$ is the radiation transfer rate with no shields ($N = 0$).

A thin aluminium sheet with an emissivity of 0.1 on both sides is placed between two very large parallel plates that are maintained at uniform temperatures $T_1 = 800\text{ K}$ and $T_2 = 500\text{ K}$ and have emissivities $\varepsilon_1 = 0.2$ and $\varepsilon_2 = 0.2$, respectively. Determine the net rate of radiation heat transfer between the two plates per unit surface area of the plates and compare the result to that without the shield.





$$(q_{12})_0 = \frac{A\sigma(T_1^4 - T_2^4)}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1} = 3.625 \text{ kW/m}^2$$

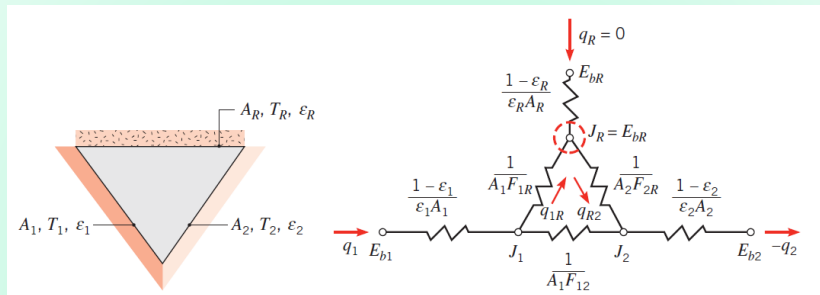
$$(q_{12})_1 = \frac{A_1\sigma(T_1^4 - T_2^4)}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \frac{1-\varepsilon_{3,1}}{\varepsilon_{3,1}} + \frac{1-\varepsilon_{3,2}}{\varepsilon_{3,2}}} = 0.806 \text{ kW/m}^2$$

Comment: Radiation heat transfer rate reduces to about one fourth of what it was as a result of placing a radiation shield between the two parallel plates.

Reradiating Surface

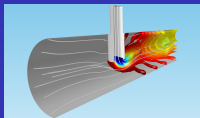


$q_i = 0$, well insulated on one side and for which convection effects may be neglected on the opposite (radiating) side.



$$q_1 = -q_2 = \frac{E_{b1} - E_{b2}}{\frac{1-\varepsilon_1}{\varepsilon_1 A_1} + \frac{1}{A_1 F_{12} + \left[\frac{1}{A_1 F_{1R}} + \frac{1}{A_2 F_{2R}} \right]^{-1}} + \frac{1-\varepsilon_2}{\varepsilon_2 A_2}}$$

Heat and Mass Transfer



Mass Transfer

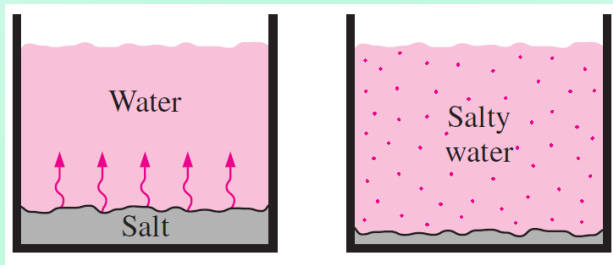
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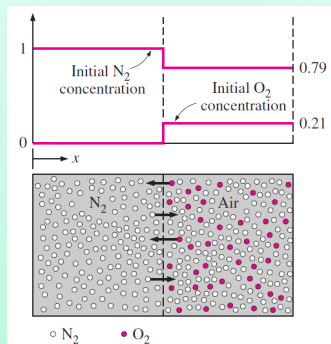
Department of Mechanical Engineering
Indian Institute of Technology Patna

- Many significant HT problems in practice involve mass transfer.
- $1/3^{\text{rd}}$ of the heat loss from a resting person is due to evaporation.
- A close resemblance between heat and mass transfer relations.





- Mass transfer requires the presence of two regions at different chemical compositions
- Movement of a chemical species from a high concentration region towards a lower concentration one relative to the other chemical species present in the medium.
- Fluid flow is due to **pressure difference**, whereas Mass transfer is due to **concentration difference**.
- No mass transfer in a homogeneous medium.



$$\begin{aligned}\text{Flow rate} &\propto (\text{Normal area})(\text{Concentration gradient}) \\ &= -DA \frac{dC}{dx}\end{aligned}$$

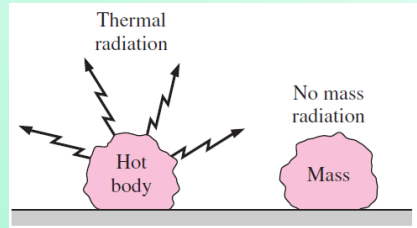
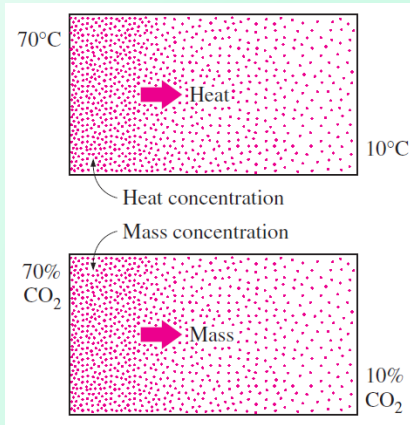
where D is the **diffusion coefficient** of the medium, a measure of how fast the mass diffuses in the medium.

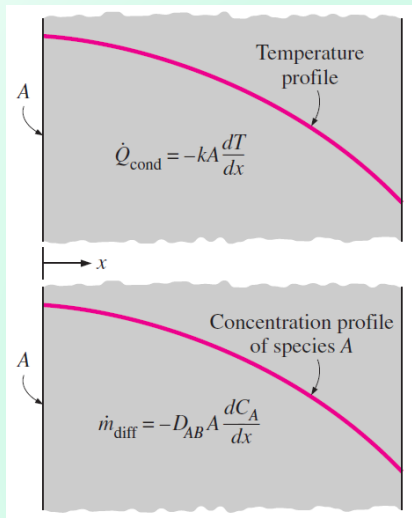


- Molecules in a gas mixture continually collide with each other.
- The collision of unlike molecules influence the rate of diffusion, as different masses have different movement.
- The diffusion process is dominated by heavier molecules
- The diffusion rates will be higher at higher temperatures.
- Molecular spacing also influences diffusion process.

$$D_{\text{Gas}} \gg D_{\text{liquids}} \gg D_{\text{solids}}$$

Analogy Between Heat and Mass Transfer





Fourier's law: $Q = -kA \frac{dT}{dx}$

Fick's law of diffusion:

Rate of mass diffusion of species A in a stationary medium in the direction x :

$$\dot{m}_{\text{diff}} = -D_{AB}A \frac{dC_A}{dx}$$

D_{AB} is mass diffusivity

C_A is conc. of species A in mixture at that location.



- The differential equations for both heat conduction and mass diffusion are of the same form.
- Therefore, the solutions of mass diffusion equations can be obtained from the solutions of corresponding heat conduction equations for the same type of boundary conditions by simply switching the corresponding coefficients and variables.

$$\frac{\partial}{\partial x} \left(\rho D_{AB} \frac{\partial w_A}{\partial x} \right) + \frac{\partial}{\partial y} \left(\rho D_{AB} \frac{\partial w_A}{\partial y} \right) + \frac{\partial}{\partial z} \left(\rho D_{AB} \frac{\partial w_A}{\partial z} \right) + \dot{m}_A''' = \frac{\partial \rho_A}{\partial t}$$

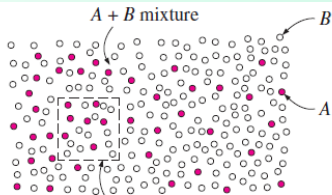
Species generation is a volumetric phenomenon, homogeneous reactions.



$$Q_{conv} = h_{conv} A_s (T_s - T_\infty)$$

$$\dot{m}_{conv} = h_{mass} A_s (C_s - C_\infty)$$

Concentration Based on Mass, Mole



$$V = V_A = V_B$$

$$m = m_A + m_B$$

$$\rho = \rho_A + \rho_B$$

$$C = C_A + C_B$$

Mass basis:

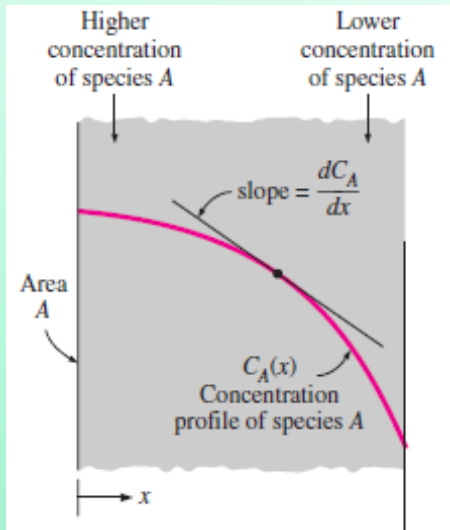
$$\rho_A = \frac{m_A}{V}, \quad \rho = \frac{m}{V}, \quad w_A = \frac{\rho_A}{\rho}$$

Mole basis:

$$C_A = \frac{N_A}{V}, \quad C = \frac{N}{V}, \quad y_A = \frac{C_A}{C}$$

Relation between them:

$$C_A = \frac{\rho_A}{M_A}, \quad w_A = y_A \frac{M_A}{M}$$





At low pressures, a gas mixture can conveniently be approximated as an ideal gas with negligible error. Ex.: A mixture of dry air and water vapor at atmospheric conditions.

Dalton's law of additive pressures:

$$P = \sum P_i$$

$$\frac{P_i}{P} = \frac{N_i R_u T / V}{N R_u T / V} = \frac{N_i}{N} = y_i$$

The **pressure fraction** of species i of an ideal gas mixture is equivalent to the **mole fraction** of that species and can be used in place of it in mass transfer analysis.

Stationary Medium Consisting of Two Species

Mass basis:

$$j_{diff,A} = \frac{\dot{m}_{diff,A}}{A} = -\rho D_{A,B} \frac{d(\rho_A/\rho)}{dx} = -\rho D_{A,B} \frac{d(w_A)}{dx} \text{ kg/s m}^2$$

Mole basis:

$$j_{diff,A} = \frac{\dot{N}_{diff,A}}{A} = -C D_{A,B} \frac{d(C_A/C)}{dx} = -C D_{A,B} \frac{d(y_A)}{dx} \text{ mol/s m}^2$$

Relations:

$$m_i = N_i M_i \implies C_i = \frac{\rho_i}{M_i} \quad \text{and} \quad C = \frac{\rho}{M}$$

$$M = \frac{m}{N} = \sum y_i M_i \quad \text{and} \quad w_i \frac{\rho_i}{\rho} = y_i \frac{M_i}{M}$$



Binary diffusion coefficients of some gases in air at 1 atm pressure (from Mills, Ref. 13, Table A.17a, p. 869)

<i>T, K</i>	<i>Binary Diffusion Coefficient, * m²/s × 10⁵</i>			
	<i>O₂</i>	<i>CO₂</i>	<i>H₂</i>	<i>NO</i>
200	0.95	0.74	3.75	0.88
300	1.88	1.57	7.77	1.80
400	5.25	2.63	12.5	3.03
500	4.75	3.85	17.1	4.43
600	6.46	5.37	24.4	6.03
700	8.38	6.84	31.7	7.82
800	10.5	8.57	39.3	9.78
900	12.6	10.5	47.7	11.8
1000	15.2	12.4	56.9	14.1
1200	20.6	16.9	77.7	19.2
1400	26.6	21.7	99.0	24.5
1600	33.2	27.5	125	30.4
1800	40.3	32.8	152	37.0
2000	48.0	39.4	180	44.8

*Multiply by 10.76 to convert to ft²/s.

$$D_{AB} \propto \frac{T^{3/2}}{P}$$

Diffusion of water vapor in air:

$$D_{\text{H}_2\text{O}, \text{air}} = 1.87 \times 10^{-10} \frac{T^{2.072}}{P}$$

$$(\text{m}^2/\text{s}), 280 \text{ K} < T < 450 \text{ K}$$

Binary diffusion coefficients of dilute gas mixtures at 1 atm
(from Barrer, Ref. 2; Geankoplis, Ref. 5; Perry, Ref. 14; and Reid et al., Ref. 15)

Substance A	Substance B	T , K	D_{AB} or D_{BA} , m^2/s	Substance A	Substance B	T , K	D_{AB} or D_{BA} , m^2/s
Air	Acetone	273	1.1×10^{-5}	Argon, Ar	Nitrogen, N ₂	293	1.9×10^{-5}
Air	Ammonia, NH ₃	298	2.6×10^{-5}	Carbon dioxide, CO ₂	Benzene	318	0.72×10^{-5}
Air	Benzene	298	0.88×10^{-5}	Carbon dioxide, CO ₂	Hydrogen, H ₂	273	5.5×10^{-5}
Air	Carbon dioxide	298	1.6×10^{-5}	Carbon dioxide, CO ₂	Nitrogen, N ₂	293	1.6×10^{-5}
Air	Chlorine	273	1.2×10^{-5}	Carbon dioxide, CO ₂	Oxygen, O ₂	273	1.4×10^{-5}
Air	Ethyl alcohol	298	1.2×10^{-5}	Carbon dioxide, CO ₂	Water vapor	298	1.6×10^{-5}
Air	Ethyl ether	298	0.93×10^{-5}	Hydrogen, H ₂	Nitrogen, N ₂	273	6.8×10^{-5}
Air	Helium, He	298	7.2×10^{-5}	Hydrogen, H ₂	Oxygen, O ₂	273	7.0×10^{-5}
Air	Hydrogen, H ₂	298	7.2×10^{-5}	Oxygen, O ₂	Ammonia	293	2.5×10^{-5}
Air	Iodine, I ₂	298	0.83×10^{-5}	Oxygen, O ₂	Benzene	296	0.39×10^{-5}
Air	Methanol	298	1.6×10^{-5}	Oxygen, O ₂	Nitrogen, N ₂	273	1.8×10^{-5}
Air	Mercury	614	4.7×10^{-5}	Oxygen, O ₂	Water vapor	298	2.5×10^{-5}
Air	Napthalene	300	0.62×10^{-5}	Water vapor	Argon, Ar	298	2.4×10^{-5}
Air	Oxygen, O ₂	298	2.1×10^{-5}	Water vapor	Helium, He	298	9.2×10^{-5}
Air	Water vapor	298	2.5×10^{-5}	Water vapor	Nitrogen, N ₂	298	2.5×10^{-5}

Note: The effect of pressure and temperature on D_{AB} can be accounted for through $D_{AB} \sim T^{3/2}/P$. Also, multiply D_{AB} values by 10.76 to convert them to ft²/s.



Ordinary diffusion mass diffusion due to the concentration gradient

Thermal diffusion mass diffusion due to the temperature gradient

Pressure diffusion mass diffusion due to the pressure gradient

Forced diffusion An external force field such as an electric or magnetic field applied on a mixture or solution can be used successfully to separate electrically charged or magnetized molecules from the mixture

Knudsen and Surface diffusion Porous solid such as silica gel are smaller than the mean free path of the gas molecules; and when the size of the gas molecules is comparable to the pore size, adsorbed molecules move along the pore walls.

All the above phenomena causing mass diffusion are usually neglected assuming the dominant driving potential is the species concentration gradient.



Fourier's law of conduction:

$$q'' = -k \frac{\partial T}{\partial x} \text{ (W/m}^2\text{)}$$

Newton's law of viscosity:

$$\tau = -\mu \frac{\partial u}{\partial x} \text{ (N/m}^2\text{)}$$

Fick's law of diffusion:

$$\dot{m}''_{diff,A} = -D_{AB} \frac{d\rho_A}{dx} \text{ (kg/s m}^2\text{)}$$

$$\bar{\dot{m}}''_{diff,A} = -D_{AB} \frac{dC_A}{dx} \text{ (kmol/s m}^2\text{)}$$

Diffusion (m^2):

Momentum (ν); Thermal (α); Mass (D_{AB})

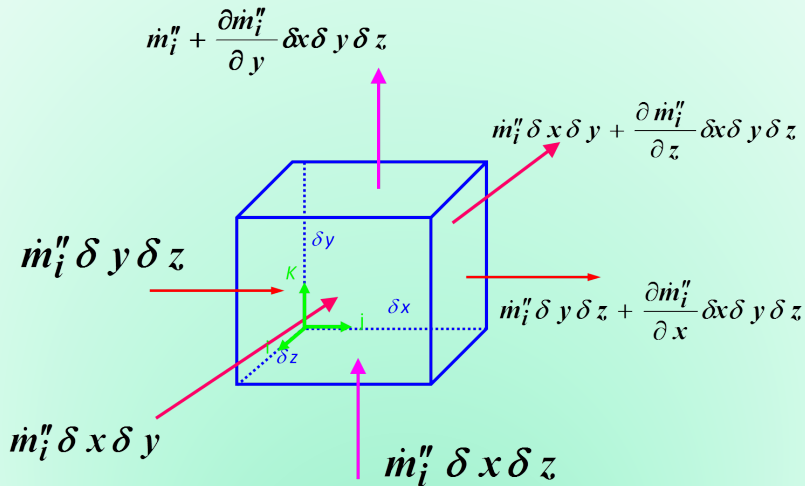


Prandtl number :
$$\text{Pr} = \frac{\nu}{\alpha} = \frac{\text{Momentum diffusivity}}{\text{Thermal diffusivity}}$$

Schmidt number :
$$\text{Sc} = \frac{\nu}{D_{AB}} = \frac{\text{Momentum diffusivity}}{\text{Mass diffusivity}}$$

Lewis number :
$$\text{Le} = \frac{\text{Sc}}{\text{Pr}} = \frac{\alpha}{D_{AB}} = \frac{\text{Thermal diffusivity}}{\text{Mass diffusivity}}$$

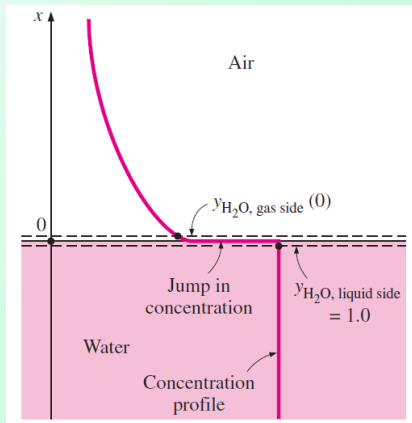
Mass Diffusion Equation





$$\dot{m}_{in} - \dot{m}_{out} + \dot{m}_g = \dot{m}_{st}$$

$$\frac{\partial}{\partial x} \left(\rho D_{AB} \frac{\partial w_A}{\partial x} \right) + \frac{\partial}{\partial y} \left(\rho D_{AB} \frac{\partial w_A}{\partial y} \right) + \frac{\partial}{\partial z} \left(\rho D_{AB} \frac{\partial w_A}{\partial z} \right) + \dot{m}_A''' = \frac{\partial \rho_A}{\partial t}$$
$$\frac{\partial}{\partial x} \left(C D_{AB} \frac{\partial y_A}{\partial x} \right) + \frac{\partial}{\partial y} \left(C D_{AB} \frac{\partial y_A}{\partial y} \right) + \frac{\partial}{\partial z} \left(C D_{AB} \frac{\partial y_A}{\partial z} \right) + \dot{N}_A''' = \frac{\partial C_A}{\partial t}$$

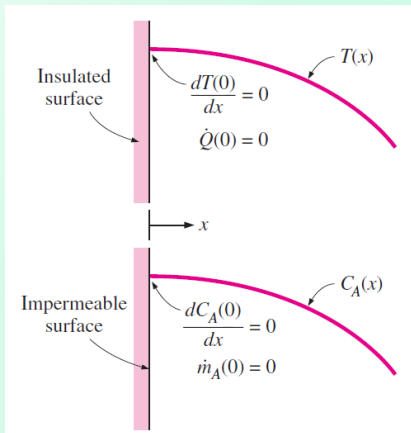


First condition:

$$y_A(0, t) = y_{A,s}$$

Second condition:

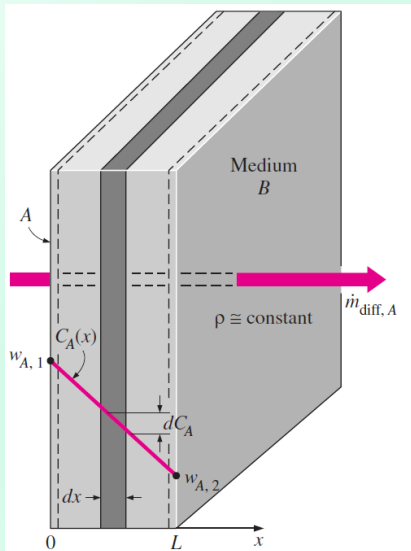
$$-CD_{AB} \left. \frac{\partial y_A}{\partial x} \right|_{x=0} = J_{A,s}^*$$



Impermeable surface:

$$\left. \frac{\partial y_A}{\partial x} \right|_{x=0} = 0$$

Steady Mass Diffusion Through a Wall

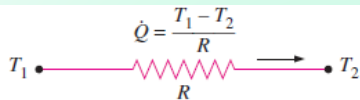


$$\dot{m}_{\text{diff},A} = j_A A = \text{constant} \quad (\text{kg/s})$$

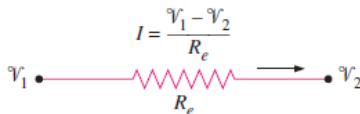
$$j_A = \frac{\dot{m}_{\text{diff},A}}{A} = -\rho D_{AB} \frac{dw_A}{dx} = \text{constant}$$

$$\dot{m}_{\text{diff},A,\text{wall}} = \frac{w_{A,1} - w_{A,2}}{L/\rho D_{AB} A}$$

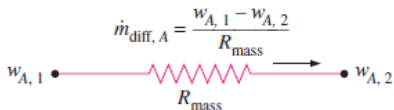
Steady Mass Diffusion Through a Wall



(a) Heat flow



(b) Current flow



(c) Mass flow

$$\dot{m}_{\text{diff},A, \text{ wall}} = \frac{w_{A,1} - w_{A,2}}{L/\rho D_{AB}A}$$

$$\dot{m}_{\text{diff},A, \text{ cyl}} = 2\pi L D_{AB} \frac{\rho_{A,1} - \rho_{A,2}}{\ln(r_2/r_1)}$$

$$\dot{m}_{\text{diff},A, \text{ sph}} = 4\pi r_1 r_2 D_{AB} \frac{\rho_{A,1} - \rho_{A,2}}{r_2 - r_1}$$